Disentanglements of maps from $2n$-space to $3n$-space.

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Abstract

It is shown that the first disentanglement of a multi-germ $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ is homotopically equivalent to a wedge of $(n+1)$-spheres and 2-spheres. For corank 1 mono-germs in the same dimensions it is shown that the second disentanglement is a homotopically equivalent to a wedge of n-spheres and circles.

Good real perturbations of such maps are investigated and it is shown for multi-germs in the case $n = 1$ with a good real perturbation that the real and complex disentanglements are homotopically equivalent.

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1 Introduction

For a finitely $\mathcal{A}$-determined map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, where $n < p$ and S is a finite set, there exists a disentanglement, the image of a stabilisation of f. This is analogous to the Milnor fibre in the $n \geq p$ and finitely $\mathcal{K}$-determined case. For mono-germs or when $p = n + 1$ it has been shown that the disentanglement of f is homotopically equivalent to a wedge of spheres (again analogously to Milnor fibres) where the spheres may be of different dimensions, see [4] and [14].

In this paper we study the situation for maps $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$. This is a natural generalisation of the case of germs of maps from surfaces to 3-space, which has been studied by Mond, in particular in [13], [14], [15], [16] and Marar and Mond, [12]. The usual generalisation of this work has been to map germs from $(\mathbb{C}^n, S)$ to $(\mathbb{C}^{n+1}, 0)$ as then the image of the map is still a hypersurface, a useful fact which is certainly not true in the case under consideration here. Many of the results in this paper are direct generalisations of the work of Mond cited above, and, as well as being interesting in their own right, they also provide insight into the topology of disentanglements of multi-germs and the real perturbations and higher disentanglements of mono-germs in general.

The $\mathbb{C}^{2n}$ to $\mathbb{C}^{3n}$ is tractable because the multiple point spaces, the closure of k-tuples of points that have the same image, of the stabilisation are particularly simple. The source is non-singular and each component is contractible, the double point space has dimension n and the triple point space has dimension zero. One of the main tools in the theory is an image computing spectral sequence introduced in [2], the columns of which depend on the alternating homology of the multiple point spaces. In our situation this means that only three columns of the sequence can contain non-trivial groups and the third column can only have a non-trivial group in the first row as the triple point space is zero-dimensional. Thus using the spectral sequence is a practical proposition.

In Section 2 the necessary definitions and notations are made. The first main result in Section 3 is that the disentanglement in the multi-germ case is homotopically equivalent to a wedge of $(n+1)$-spheres and 2-spheres. This corresponds to what happens in general for mono-germs, see Proposition 4.24 of [4], and thus provides evidence for multi-germ disentanglements being homotopically equivalent to a wedge of spheres.

In a disentanglement we can look at the closure of the set of points with two or more preimages, this is called the second disentanglement. One can generalise to the kth disentanglement: the closure of the set of points with k or more preimages. When f is a corank 1 mono-germ and $p = n+1$ it has been shown in [7] that the kth disentanglement is homotopically equivalent to a wedge of $(n-k+1)$-spheres, following [2] where it was shown that they have reduced rational cohomology in dimension $n-k+1$. In the situation of maps from $2n$-space to $3n$-space we again have to restrict to corank 1 mono-germs to treat the second disentanglement. This is also homotopically equivalent to a wedge of spheres, this time of dimensions n and 1. The topology of the the higher disentanglements (i.e. $k > 1$) is difficult to calculate in general and this is the first occasion on which we have a non-trivial result for $p > n+1$.

We do not formally treat the third disentanglement as this is just the set of triple points.

If one restricts the disentanglement map to the real part of the source then one gets real perturbations. For maps $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ Mond defines in [17] a real perturbation to be good if the rank of the nth integer homology group of the image of the real perturbation is equal to the rank of the nth integer homology group of the disentanglement. In Section 4 this notion is generalised and it is shown in the $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ situation that if the inclusion of the real disentanglement into the complex one produces an isomorphism on integer homology groups then the spaces are in fact homotopically equivalent. This is applied in the $n = 1$ case to show that if f has a good real perturbation then the real and complex disentanglements are homotopically equivalent. This improves a result in [12] (for $n = 1$) which states that the existence of a good real perturbation produces an isomorphism on the rational cohomology groups.

In [12] it is shown that each member in the H_k family of singularities given by $f(x, y) = (x, y^3, xy + y^{3k-1})$ has a good real perturbation. This family is generalised to the $2n$-space to $3n$-space situation and it is shown that not only do the maps have good real perturbations but the real and complex rth disentanglements for $r = 1, 2, 3$ are in fact homotopically equivalent as well.

The final section is concerned with calculating the number of triple points in a disentanglement of a quasihomogeneous corank 1 map germ using weights and degrees.

2 Notation and definitions

2.1 Multiple point spaces

We first fix some notation for a finite and proper continuous map $f : X \to Y$.

Definition 2.1 *The kth multiple point space, $D^k(f)$, of the map $f : X \to Y$ is the set of points,*

$$D^k(f) := \text{closure}\{(x_1, \dots, x_k) \in X^k | f(x_1) = \dots = f(x_k), x_i \neq x_j \textit{ for } i \neq j\}.$$

The group of permutations on k objects, denoted S_k, acts on $D^k(f)$ in the natural way. For each k there exists a map $\varepsilon_k : D^k(f) \to D^{k-1}(f)$ given by restriction of the natural projection map $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k-1})$. We also have a map $\varepsilon^k : D^k(f) \to Y$ given by $\varepsilon^k(x_1, \ldots, x_k) = f(x_1)$.

Define $D^2_1(f)$ to be the image of ε_2 in $D^1(f)$. The double point space of ε_2 is homeomorphic, by an S_k-equivariant map, to the triple point space of f and the kth multiple point spaces, $k \geq 2$, for the map $D^2_1(f)$ to Y are the same as for f. This is part of a general phenomenon, see [2].

Definition 2.2 *The set of points $(x_1, \ldots, x_k) \in D^k(f)$ such that $x_i = x_j$ for some $i \neq j$ is called the diagonal of $D^k(f)$.*

Definition 2.3 *The kth image multiple point space of f, denoted $M_k(f)$, is defined to be*

$$M_k(f) := \text{closure}\{y \in Y | \#f^{-1}(y) \geq k\}.$$

If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a corank 1 multi-germ then the multiple point spaces of f can be defined using adapted Vandermonde determinants, see [11]. For finitely $\mathcal{A}$-determined maps this definition coincides with the one given above as long as $\dim D^k(f) > 0$ or f is stable. When $\dim D^k(f)$ is expected to be 0 then $D^k(f)$ can be empty while the Vandermonde definition gives a space lying in the diagonal. It will be clear from context which definition is being used.

The Vandermonde definition leads to the following: If f is finitely $\mathcal{A}$-determined then the multiple point spaces are isolated complete intersection singularities (of dimension $nk - p(k-1)$) and vice versa. The multiple point spaces for the disentanglement map (see section 2.4) are Milnor fibres of these singularities. See [11] for details.

2.2 Alternating homology

Suppose S_k is the group of permutations on k objects and $\text{sign} : S_k \to \{1, -1\}$ is the usual sign representation of S_k.

Definition 2.4 *Let M be a module upon which S_k acts, then*

$$M^{alt} := \{m \in M | h(m) = \text{sign}(h)m \textit{ for all } h \in H\},$$

is the alternating module of M with respect to S_k.

Definition 2.5 *The group S_k acts cellularly on a cellular complex if S_k takes cells to cells and if the point of a cell is fixed by an element of S_k then the whole cell is fixed by that element.*

Definition 2.6 *Suppose Y is a space upon which S_k acts cellularly and let $C_*(Y)$ denote the cellular chain complex of Y. Define the alternating homology of Y with respect to S_k and the coefficient group G to be*

$$H^{alt}_i(Y; G) := H_i((C_*(Y))^{alt} \otimes G).$$

Define an operator $\text{Alt}_{\mathbb{Z}} : M \to M$ by

$$\text{Alt}_{\mathbb{Z}} = \sum_{h \in S_k} \text{sign}(h)h.$$

Once can take the homology generated by $\text{Alt}_{\mathbb{Z}} C_*(Y)$, (call it the *alternated homology of* Y), and we have the following theorem.

Theorem 2.7 *Suppose Y is an S_k-invariant subset of X^k and S_k acts on X^k through permutation of copies of X then*

$$H_i^{alt}(Y;G) \cong H_i((\mathrm{Alt}_{\mathbb{Z}}\, C_*(Y)) \otimes G),$$

for all i.

For a proof of this non-trivial theorem see [5]. We can use the operator $\mathrm{Alt}_{\mathbb{Z}}$ to study alternating homology; this will be particularly useful for the zeroth alternating homology groups.

Definition 2.8 *Two complexes K and L are S_k-homotopically equivalent if there exist two equivariant maps $\alpha : K \to L$ and $\beta : L \to K$, such that the compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to the identities in the class of S_k-equivariant maps.*

Complexes that are S_k-homotopically equivalent have the same alternating homology.

One has another, weaker, condition.

Definition 2.9 *Suppose G is a group, then two topological spaces X and Y are G-homology equivalent if there exists a continuous map $f : X \to Y$ such that $f_* : H_i(X;G) \to H_i(Y;G)$ is an isomorphism for all i. Such a map is called a G-homology equivalence.*

2.3 Image computing spectral sequence

Now we shall state a general version of the image computing spectral sequence which was introduced in [2] and further developed in [1] and [5].

Theorem 2.10 *Suppose that $f : X \to Y$ is a proper and finite analytic map between compact subanalytic spaces and that $\widetilde{X}$ is a subset of X. Then, for any coefficient group G, there exists a spectral sequence*

$$E^1_{p,q}(f, f|\widetilde{X}) = H_q^{alt}(D^{p+1}(f), D^{p+1}(f|\widetilde{X}); G) \implies H_*(f(X), f(\widetilde{X}); G).$$

The differential is induced from the map $\varepsilon_k : D^k(f) \to D^{k-1}(f)$.

Generally in this paper we take $\widetilde{X} = \emptyset$.

To produce this theorem one uses a *semi-simplicial resolution of the image.* This is a space for which a point of the image is represented by a $(k-1)$-simplex if the point has k preimages. This construction will be used to describe the homotopy type of the disentanglements.

The resolution can be constructed by two methods, for the general method see [5], we shall only describe the case in which the image is embeddable in $\mathbb{R}^N$ for some N, as this will be true for all our images.

Let m be the maximal number of preimages of a point of f. Consider an embedding of X into some $\mathbb{R}^N$ so that any m distinct points of X do not lie in an $(m-2)$-dimensional affine plane. Let $x_1, \ldots, x_k \in X$ be the preimages of a point $y \in Y$ and consider in $y \times \mathbb{R}^N$ the closed $(k-1)$-dimensional simplex with vertices $(y, x_1), \ldots, (y, x_k)$. The union W of all such simplices for all points of Y is the semi-simplicial resolution of Y.

The filtration of Y given by defining Y_k to be the union of simplices of dimension less than k, with $Y_1 = X$, leads to the spectral sequence.

In the cases under discussion in this paper the image of a map will be homotopically equivalent to its semi-simplicial resolution.

2.4 Disentanglements

Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, is a finitely $\mathcal{A}$-determined complex analytic map-germ. Then, there exists

(i) a 1-parameter unfolding of f, $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$, such that $F(x, 0) = f(x)$,

(ii) a closed ball B_ϵ in $\mathbb{C}^p$ centred at 0 and of radius ϵ,

(iii) and a closed disc D_δ in $\mathbb{C}$ centred at 0 and of radius δ,

such that for a proper representative of F,

(i) the map $f_t := F|F^{-1}(B_\epsilon \times \{t\}) : F^{-1}(B_\epsilon \times \{t\}) \to B_\epsilon \times \{t\}$ is topologically stable for all $t \in D_\delta - \{0\}$,

(ii) the image of f_0 is Whitney stratified so that every $(2p-1)$-sphere $S_{\epsilon'} \subset B_\epsilon$ of radius ϵ' centred at 0 is stratified transverse to it for all $0 < \epsilon' \leq \epsilon$,

(iii) the map f_t is topologically right-left equivalent to $f_{t'}$ for all $t, t' \in D_\delta - \{0\}$.

Details can be found in [10] and [2].

Definition 2.11 *The ball B_ϵ is called a Milnor ball and ϵ is called a Milnor radius. The disc D_δ is called a Milnor disc.*

The next definition is important in the rest of the paper.

Definition 2.12 *The set $M_k(f_t)$ for $t \neq 0$ is called the kth disentanglement of f and is denoted* $\mathrm{Dis}_k(f)$.

We shall call f_t for $t \neq 0$ the disentanglement map. The image computing spectral sequence of Theorem 2.10 exists for this map.

The disentanglement of f is essentially unique in that for any other unfolding for which a disentanglement can be constructed then the two disentanglements are homeomorphic. Again see [10] and [2] for details.

3 Topology of the first disentanglement

3.1 Homology of the first disentanglement

For maps $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ we will investigate the image computing spectral sequence for the disentanglement map and show that the first disentanglement has free abelian homology in dimensions $n+1$ and 2.

Theorem 3.1 *Suppose that $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a finitely $\mathcal{A}$-determined multi-germ and that f_t is the disentanglement map. Let $T(f_t)$ denote the number of triple points in the image of f_t and let m be the number of elements of S.*

Then, the image computing spectral sequence $E_{p,q}$ for f_t degenerates at E^2 and has the following form:

$$E^1_{0,0} \cong \mathbb{Z}^m,\ E^1_{1,0} \cong \mathbb{Z}^{m(m-1)/2},\ E^1_{2,0} \cong \mathbb{Z}^{T(f_t)},\ E^1_{1,n} \cong \mathbb{Z}^{\mu_1},$$

for some μ_1, and

$$E^2_{0,0} \cong \mathbb{Z},\ E^2_{1,n} \cong \mathbb{Z}^{\mu_1},\ E^2_{2,0} \cong \mathbb{Z}^{\mu_2},$$

where $\mu_2 = T(f_t) - \frac{1}{2}(m-2)(m-1)$. All other terms are zero.

Proof. As f is finitely $\mathcal{A}$-determined there exists a 1-parameter unfolding $F : (\mathbb{C}^{2n} \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^{3n} \times \mathbb{C}, 0 \times 0)$ so that $F(x,0) = f(x)$ and $f_t(x) = F(x,t)$ gives the disentanglement map. Through a choice of a good representative we get a map $F : U \to V$ such that the image of F is contractible and we get a fibration of the form of Propositions 3.20 and 3.21 of [4], i.e. maps $g_k : D^k(F) \to D_\delta$, with D_δ a sufficiently small Milnor disc, with fibres $D^k(f_t)$ over $D_\delta - \{0\}$.

As f_t is topologically stable $\dim_{\mathbb{C}} D^k(f_t) = 2nk - 3n(k-1) = n(3-k)$, where this is non-negative. By Corollary 3.2 of [4] $H_i^{alt}(D^k(f_t); \mathbb{Z})$ is free abelian for $i = n(3-k)$, $k \leq 3$.

Theorems 3.13 and 3.30 of [4] give $H_i^{alt}(D^k(F), D^k(f_t); \mathbb{Z}) = 0$ for $i \leq n(3-k) = \dim_{\mathbb{C}} D^k(f_t)$. As $D^k(F)$ contracts equivariantly onto $D^k(F) \cap (\varepsilon^k)^{-1}(0)$ for all k the alternating homology of these multiple point spaces is the same as for the map $g : \{m \text{ points}\} \to \{1 \text{ point}\}$, and so by Example 2.5 of [4] we find,

$$\begin{aligned} H_0^{alt}(D^1(F); \mathbb{Z}) &= \mathbb{Z}^m, \\ H_0^{alt}(D^2(F); \mathbb{Z}) &= \mathbb{Z}^{m!/(m-2)!2!}, \text{ and} \\ H_0^{alt}(D^3(F); \mathbb{Z}) &= \mathbb{Z}^{m!/(m-3)!3!}. \end{aligned}$$

From these facts we can calculate that $E^1_{p,q}$ for f_t has the stated form.

For E^2 we need only compare the $E^1_{*,0}$ complexes for F and f_t in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \leftarrow & H_0^{alt}(D^1(f_t); \mathbb{Z}) & \leftarrow & H_0^{alt}(D^2(f_t); \mathbb{Z}) & \leftarrow & H_0^{alt}(D^3(f_t); \mathbb{Z}) & \leftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \leftarrow & H_0^{alt}(D^1(F); \mathbb{Z}) & \leftarrow & H_0^{alt}(D^2(F); \mathbb{Z}) & \leftarrow & H_0^{alt}(D^3(F); \mathbb{Z}) & & \end{array}$$

The bottom row is exact except at the first position where it has homology equal to $\mathbb{Z}$, this is because the image of F is contractible and $E^1_{p,q}(F) = 0$ for $q > 0$.

The first and second vertical arrows are isomorphisms and the third is surjective so

$$E^2_{0,0}(f_t) = E^2_{0,0}(F) = \mathbb{Z}, \ E^2_{1,0}(f_t) = E^2_{1,0}(F) = 0,$$

and $E^2_{2,0}(f_t)$ is free abelian. Thus $E^2(f_t)$ has the form of the theorem statement.

That $\mu_1 = T(f_t) - \frac{1}{2}(m-2)(m-1)$ follows from a simple calculation involving Euler characteristics applied to the complex

$$0 \leftarrow \mathbb{Z}^m \leftarrow \mathbb{Z}^{m(m-1)/2} \leftarrow H_0^{alt}(D^3(f); \mathbb{Z}) \cong \mathbb{Z}^{T(f_t)} \leftarrow 0$$

which has homology $\mathbb{Z}, 0, \mathbb{Z}^{\mu_2}$. $\square$

Corollary 3.2 *Suppose that $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a finitely $\mathcal{A}$-determined multi-germ. Then the reduced integer homology of the first disentanglement of f is free abelian in dimensions $n+1$ and 2 and zero in all other dimensions.*

Proof. The spectral sequence for f_t calculates the homology of $\mathrm{Dis}_1(f)$. Hence, from the theorem $H_0(\mathrm{Dis}_1(f); \mathbb{Z}) \cong E^2_{0,0} \cong \mathbb{Z}$. If $n = 1$ then $H_2(\mathrm{Dis}_1(f); \mathbb{Z}) \cong \mathbb{Z}^{\mu_1 + \mu_2}$, else $H_2(\mathrm{Dis}_1(f); \mathbb{Z}) \cong E^2_{2,0} \cong \mathbb{Z}^{\mu_2}$ and $H_{n+1}(\mathrm{Dis}_1(f); \mathbb{Z}) \cong E^2_{1,n} \cong \mathbb{Z}^{\mu_2}$. $\square$

3.2 Homotopy type of the first disentanglement

We will show that the first disentanglement of a finitely $\mathcal{A}$-determined multi-germ $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ is homotopically equivalent to a wedge of spheres of dimension $n+1$ and 2. For mono-germs this has been proved in Theorem 4.24 of [4]. The case $n = 1$ was effectively proved earlier in [14], the statement there is made for mono-germs but the proof works equally well for the multi-germ case.

We begin with a lemma.

Lemma 3.3 *Suppose $g : U \to Y$ is a finite and proper surjective simplicial map of compact spaces, with U contractible and $D^3(g)$ is a finite set of points or curves with each component containing a point in the diagonal and $D^4(f) = \emptyset$. Suppose for the image computing spectral sequence we have*

$$E^1_{0,0} \cong \mathbb{Z}, \ E^1_{1,n} \cong \mathbb{Z}^{\mu_1}, \ E^1_{2,0} \cong \mathbb{Z}^{\mu_2},$$

for $n \geq 1$ and that all other terms are zero. Then Y is homotopically equivalent to a wedge of μ_1 $(n+1)$-spheres and μ_2 2-spheres.

Proof. The proof uses some theorems from [4] that were stated for complex analytic maps, however they can be adapted to work in the current context.

From the semi-simplicial resolution of the image we have a filtration

$$Y_0 = \emptyset, U = Y_1 \subset Y_2 \subset Y_3,$$

such that Y_3, the semi-simplicial resolution of Y, is homotopically equivalent to Y, see Section 2.3. As $H^{alt}_0(D^2(g);\mathbb{Z}) = 0$ we get from Corollary 4.19 of [4] that Y is simply connected.

If $n = 1$ then Y is a simply connected complex with free abelian integer homology in dimension 2 (by Corollary 3.2) and hence is homotopically equivalent to a wedge of 2-spheres. Thus assume $n > 1$.

From the construction of the image computing spectral sequence, see [1] or [5], we get that $H^{alt}_{i-k+1}(D^k(g);\mathbb{Z}) \cong H_i(Y_k, Y_{k-1};\mathbb{Z})$. Since $H^{alt}_0(D^2(g);\mathbb{Z}) = 0$ we get $\pi(Y_2, Y_1) = 0$ by Proposition 4.22 of [4]. By $E^1_{1,i} = H^{alt}_i(D^2(g);\mathbb{Z}) \cong H_{i+1}(Y_2, Y_1;\mathbb{Z})$ we have that $H_{n+1}(Y_2, Y_1;\mathbb{Z}) \cong \mathbb{Z}^{\mu_1}$ and all other relative groups are trivial. Thus by Lemma 4.23 of [4] and the fact that $Y_1 = U$ is contractible we deduce that $Y_2 \simeq Y_1 \vee (\vee_{\mu_1} S^{n+1}) \simeq \vee_{\mu_1} S^{n+1}$.

Similarly we have $H_2(Y_3, Y_2) \cong \mathbb{Z}^{\mu_2}$ and all other groups trivial, Y_2 is n-connected (so at least 1-connected as $n > 1$) and $\pi_1(Y_3, Y_2) = 0$ as we are adding 2-cells to Y_2 to get Y_3. So again applying Lemma 4.23 of [4] we find that $Y \simeq Y_3 \simeq (\vee_{\mu_1} S^{n+1}) \vee (\vee_{\mu_2} S^2)$. □

We now state one of the main theorems of this paper. The proof utilises a novel trick that cannot immediately be applied to the general multi-germ case but which may still be useful in other contexts.

Theorem 3.4 *Suppose that $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a finitely $\mathcal{A}$-determined complex analytic multi-germ. Then, the first disentanglement, $\mathrm{Dis}_1(f)$, is homotopically equivalent to a wedge of spheres of dimension $n+1$ and 2. The number of 2-spheres arising from the triple point space of the disentanglement is given as μ_2 in Theorem 3.1.*

Proof. In the following proof let f denote the disentanglement map f_t, let Y be the image of f and let $U = U_1 \amalg U_2 \amalg \cdots \amalg U_m$ be the source of f. Effectively by induction we will construct a map $\widetilde{f}$ for which the natural morphism $E^1_{p,q}(f) \to E^1_{p,q}(\widetilde{f})$ is an isomorphism for all $q > 0$, $E^2_{p,q}(f) \to E^2_{p,q}(\widetilde{f})$ is an isomorphism for all p and q, and the image of $\widetilde{f}$ has the same homotopy type as Y. At each step the main difference is that the complex $E^1_{p,0}$ is simpler than for the preceding map. We continue until we produce a map that satisfies the assumptions of Lemma 3.3.

Suppose that $(x_1, x_2, x_3) \in D^3(f)$ is a triple point such that $\varepsilon_{3*}(\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2, x_3))$ is non-zero in $H^{alt}_0(D^2(f);\mathbb{Z})$ and that each x_i is in a distinct component of U.

Then create $\widetilde{f} : \widetilde{U} \to \widetilde{Y}$ where $\widetilde{U}$ is equal to U but with the three points x_1, x_2 and x_3 in U connected by intervals to a new point x. The set $\widetilde{Y}$ is Y with an interval attached to the point $y = f(x_1) = f(x_2) = f(x_3)$. Then let $\widetilde{f}$ be the same as f but with the intervals x_i to x all mapped in the obvious way to the interval beginning at y.

Since f is the restriction of $\widetilde{f}$ to a subset of $\widetilde{U}$ we have a morphism of spectral sequences $E^1_{p,q}(f) \to E^1_{p,q}(\widetilde{f})$. The multiple point spaces $D^1(\widetilde{f})$, $D^2(\widetilde{f})$ and $D^3(\widetilde{f})$ are the same as those of f but with previously unconnected components attached by intervals to x, (x,x) and (x,x,x) respectively. Thus $E^1_{p,q}(f) \to E^1_{p,q}(\widetilde{f})$ is an isomorphism for all $q > 0$.

Since $\widetilde{Y}$ is Y with an interval attached by only one end, they are homotopically equivalent and so the inclusion $Y \hookrightarrow \widetilde{Y}$ is a $\mathbb{Z}$-homology equivalence. This then implies as both sequences collapse at E^2 that $E^2_{p,q}(f) \to E^2_{p,q}(\widetilde{f})$ is an isomorphism for all p and q.

By doing this process repeatedly for triple points in distinct components of the (new) source we can produce a surjective map $f' : U' \to Y'$ such that $E^1_{p,q}(f) \to E^1_{p,q}(f')$ is an isomorphism for all $q > 0$, $E^2_{p,q}(f) \to E^2_{p,q}(f')$ is an isomorphism for all p and q and Y' is homotopically equivalent to Y.

We now wish to create a map that has $E^1_{0,0} = \mathbb{Z}$ and $E^1_{1,0} = 0$. Suppose that (x_1, x_2) is a point such that $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2)$ is non-zero, i.e. is a generator of $H_0^{alt}(D^2(f'); \mathbb{Z})$.

In a similiar way to the case of triple points one can construct a map $\widetilde{f}$ by joining the points x_1 and x_2 to a new point x. Again this map has the same properties as f' at E^2 and the image has the same homotopy type as Y'. Carry out this process for all generators of $H_0^{alt}(D^2(\widetilde{f}); \mathbb{Z})$.

Thus we produce a map $\widetilde{f} : \widetilde{U} \to \widetilde{Y}$ which has

$$E^1_{p,q}(\widetilde{f}) = \begin{cases} \mathbb{Z} & \text{for } (p,q) = (0,0), \\ \text{free abelian} & \text{for } (p,q) = (1,n) \text{ and } (2,0), \\ 0 & \text{otherwise.} \end{cases}$$

The set $\widetilde{U}$ is contractible as each original component of U was a contractible ball and $\widetilde{U}$ has all these components attached to a point.

The map has the form of Lemma 3.3 and so $Y \simeq \widetilde{Y}$ is homotopically equivalent to a wedge of spheres of dimension $n+1$ and 2. □

4 Topology of the second disentanglement

In this section we investigate the topology of the second disentanglement in the case that f is a corank 1 mono-germ. When $n = 1$ the rational cohomology has been studied in [7] (although admittedly this is a rather simple situation as the second disentanglement is a curve for $\mathbb{C}^2$ to $\mathbb{C}^3$).

The second disentanglement will be shown to be homotopically equivalent to the quotient space $D^2(f_t)/S_2$ union a finite number of hexagons and so we first look at the topology of $D^2(f_t)/S_2$.

Theorem 4.1 *Suppose $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a finitely $\mathcal{A}$-determined map-germ and that f_t denotes the disentanglement map. Then,*

(i)

$$H_i(D^2(f_t)/S_2; \mathbb{Z}) \cong \begin{cases} H_i(D^2(f_t); \mathbb{Z}), & \textit{for } i \neq n, \\ H_n(D^2(f_t); \mathbb{Z})/id_* H_n^{alt}(D^2(f_t); \mathbb{Z}), & \textit{for } i = n, \end{cases}$$

where $id : D^2(f_t) \to D^2(f_t)$ is the identity map.

(ii) *If $D^2(f_t)$ is connected then the natural map $\pi_1(D^2(f_t)) \to \pi_1(D^2(f_t)/S_2)$ is surjective.*

Proof. Consider the natural map $g : D^2(f_t) \to D^2(f_t)/S_2$. This map has no triple points so the projection $\varepsilon_2 : D^2(g) \to D^1(g) = D^2(f_t)$ is injective. As it is also proper and surjective it is a homeomorphism onto $D^1(g)$. It is also equivariant. Thus the natural map $H_i^{alt}(D^2(g);\mathbb{Z}) \to H_i^{alt}(D^2(f_t);\mathbb{Z})$ is an isomorphism for all i.

Now consider the image computing spectral sequence for g,

$$\begin{aligned} E^1_{0,i} &\cong H_i(D^2(f_t);\mathbb{Z}), \text{ and} \\ E^1_{1,n} &\cong H_n^{alt}(D^2(f_t);\mathbb{Z}) = \mathbb{Z}^\mu, \text{ by Proposition 4.6 of [4].} \end{aligned}$$

All other terms are zero. The only possible non-trivial differential is $d : E^1_{1,n} \to E^1_{0,n}$. As this arises from the homeomorphism $\varepsilon_2 : D^2(g) \to D^1(g)$ we have

$$E^2_{0,n} \cong H_n(D^2(f_t);\mathbb{Z})/\varepsilon_{2*}H_n^{alt}(D^2(f_t);\mathbb{Z}).$$

The sequence collapses at E^2 since only the terms $E^2_{0,i}$ are non-trivial. We then find that

$$H_i(D^2(f_t)/S_2;\mathbb{Z}) \cong H_i(g(D^2(f_t));\mathbb{Z}) \cong E^2_{0,i}.$$

Thus we have proved (i).

As $D^2(f_t) = D^2(g)$ is connected and $H_0^{alt}(D^2(g);\mathbb{Z}) = 0$ we have by Corollary 4.19 of [4] that the natural map $\pi_1(D^2(f_t)) = \pi_1(D^1(g)) \to \pi_1(g(D^2(f_t))) = \pi_1(D^2(f_t)/S_2)$ is surjective. □

Corollary 4.2 *Suppose that f has corank 1, then $D^2(f_t)/S_2$ is homotopically equivalent to a wedge of n-spheres, the number of which is equal to the rank of the symmetric part of $H_n(D^2(f_t);\mathbb{Z})$.*

Proof. As f is corank 1, $D^2(f_t)$ is the Milnor fibre of the n-dimensional isolated complete intersection singularity $D^2(f)$, see [11], and so it is homotopically equivalent to a wedge of μ n-spheres. Thus by part (i) of the theorem we have

$$H_i(D^2(f_t)/S_2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^\mu/\varepsilon_{2*}H_0^{alt}(D^2(f_t);\mathbb{Z}), & \text{for } i = n, \\ \mathbb{Z}, & \text{for } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

As $H_0^{alt}(D^2(f_t);\mathbb{Z})$ is the alternating part of $H_n(D^2(f_t);\mathbb{Z})$ by Theorem 2.1.2 of [1], we get that $H_n(D^2(f_t)/S_2;\mathbb{Z})$ must be isomorphic to the symmetric part of $H_n(D^2(f_t);\mathbb{Z})$.

If $n = 1$ then $D^2(f_t)/S_2$ is homotopically equivalent to a compact curve and as $D^2(f_t)$ is connected this implies the homotopy result in this case.

So now assume that $n > 1$. Then $D^2(f_t)$ is simply connected and hence by part (ii) of the theorem $D^2(f_t)/S_2$ is simply connected. A simply connected n-dimensional CW-complex with non-reduced integer homology free abelian in dimension n is homotopically equivalent to a wedge of n-spheres. □

Remark 4.3 *It would be interesting to find a map (with corank greater than one) for which $D^2(f_t)$ is not homotopically equivalent to a wedge of spheres as this could possibly be used to show that its second disentanglement was not homotopically equivalent to a wedge of spheres.*

We now prove another of the main theorems of this paper. This gives us non-trivial examples of the behaviour of second disentanglements when the target dimension of the germ is greater than the source dimension plus 1.

Theorem 4.4 *Suppose that $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a finitely $\mathcal{A}$-determined corank 1 map-germ. Then, $\mathrm{Dis}_2(f)$ is homotopically equivalent to the wedge of $\frac{1}{2}(\mu(D^2(f)) - \mu(D^2(f)|H))$ n-spheres and $(\mu(D^3(f)) + 1)/3 = 2T(f_t)$ circles. Here H is the diagonal in $D^2(f)$ and μ denotes Milnor number.*

Proof. As stated in Section 2 the images of maps are homotopically equivalent to their semi-simplicial resolutions.

Outside the triple points the map $g : D^2(f_t) \to D^2_1(f_t) \to M_2(f_t) = \mathrm{Dis}_2(f)$ is the same as the map $D^2(f_t) \to D^2(f_t)/S_2$ as all the map g is doing is identifying (x_1, x_2) with (x_2, x_1) for all $(x_1, x_2) \in D^2(f_t)$.

Now, $D^2_1(f_t)$ is homotopically equivalent to $D^2(f_t)$ with intervals connecting (x_1, x_2) to (x_1, x_3), (x_2, x_1) to (x_2, x_3) and (x_3, x_1) to (x_3, x_2) for every triple point (x_1, x_2, x_3). This is because $\varepsilon_2 : D^2(f_t) \to D^2_1(f_t)$ is surjective and its double point space is homeomorphic to $D^3(f_t)$. Thus, the semi-simplicial resolution of $D^2_1(f_t)$ is just $D^2(f_t)$ with points arising from triples connected with an interval as just explained. One can now connect every point (a, b) to (b, a) in this semi-simplicial resolution of $D^2_1(f_t)$. For each triple point one can attach a hexagon along the edges

$$(x_1, x_2) \to (x_2, x_1) \to (x_2, x_3) \to (x_3, x_2) \to (x_3, x_1) \to (x_1, x_3) \to (x_1, x_2).$$

Call this new space W. Note that W is homeomorphic to the semi-simplicial resolution of $D^2(f_t)/S_2$ union some hexagons, one for each triple and that if one contracts the edges (x_1, x_2) to (x_1, x_3), (x_2, x_1) to (x_2, x_3) and (x_3, x_1) to (x_3, x_2) in each hexagon then one gets the semi-simplical resolution of $M_2(f_t) = \mathrm{Dis}_2(f)$.

In summary,

$$\text{resolution of } D^2(f_t)/S_2 \cup \{\text{hexagons}\} \simeq W \simeq \text{resolution of } \mathrm{Dis}_2(f) \simeq \mathrm{Dis}_2(f).$$

Through contracting the intervals connecting (x_1, x_2) to (x_2, x_1) for all $(x_1, x_2) \in D^2(f_t)$ in the semi-simplicial resolution of $D^2(f_t)/S_2$ we produce $D^2(f_t)/S_2$. Then, any hexagon attached to the resolution of $D^2(f_t)/S_2$ will have the edges (x_1, x_2) to (x_2, x_1) etc, contracted. Thus we will have a space homotopically equivalent to $D^2(f)/S_2$ with triangles attached at the vertices $(x_1, x_2) = (x_2, x_1)$, $(x_1, x_3) = (x_3, x_1)$ and $(x_3, x_2) = (x_2, x_3)$ for every triple point (x_1, x_2, x_3).

By Corollary 4.2 $D^2(f_t)/S_2$ is homotopically equivalent to a wedge of n-spheres, the number, μ say, being the rank of the symmetric part of $H_n(D^2(f_t); \mathbb{Z})$.

If $n > 1$ then $D^2(f_t)/S_2$ is simply connected and so adding the edges of a triangle gives a wedge of three circles. Adding the interior of the triangle causes one of the edges to be contracted onto the other two edges. Hence the wedge is now of two circles. This can be done for every triple point so that we get $D^2(f_t)/S_2$ wedged with $2T(f_t)$ circles.

If $n = 1$ then $D^2(f_t)/S_2$ is homotopically equivalent to a wedge of circles. Adding $3T(f_t)$ 1-cells (the edges of triangles) and $T(f_t)$ 2-cells (the triangles) will alter the Euler characteristic by $2T(f_t)$ and contracting one edge of the triangle onto the other two will give a wedge of spheres.

Hence $\mathrm{Dis}_2(f)$ is homotopically equivalent to μ n-spheres and $2T(f_t)$ circles. That $\mu = \frac{1}{2}(\mu(D^2(f)) - \mu(D^2(f)|H))$ follows from the fact that the rank of the alternating part of $H_n(D^2(f_t); \mathbb{Z})$ is equal to $\frac{1}{2}(\mu(D^2(f)) + \mu(D^2(f)|H))$, see [9] for example, and that the sum of the ranks of the alternating and symmetric parts is $\mu(D^2(f))$. The number of triple points is equal to $(\#D^3(f_t))/6$ which is equal to $(\mu(D^3(f)) + 1)/6$. □

Remark 4.5 *The image computing spectral sequence for the restriction $f_t|D^2_1(f_t) : D^2_1(f_t) \to \mathrm{Dis}_2(f)$ is interesting because it exhibits a non-trivial second differential. For simplicity consider the case $n > 1$, then the differential $d_2 : E^2_{2,0} \cong \mathbb{Z}^{T(f_t)} \to$*

$E^2_{0,1} \cong \mathbb{Z}^{3T(f_t)}$ *must be non-zero or else all of* $E^2_{0,1}$ *survives to infinity and thus* $\mathrm{Dis}_2(f)$ *has its first homology group isomorphic to* $\mathbb{Z}^{3T(f_t)}$ *and not* $\mathbb{Z}^{2T(f_t)}$. *A similar situation arises in the case of maps where* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, *see [7].*

It would be interesting to get a proof of this fact about differentials from some other principle rather than working backwards as we have just done since this would be useful in the study of higher order disentanglements of maps $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *with* $n < p$.

5 Good real perturbations

5.1 General theorems

Throughout this section we assume that f is the complexification of a finitely $\mathcal{A}$-determined real multi-germ $f' : (\mathbb{R}^{2n}, S) \to (\mathbb{R}^{3n}, 0)$, $n \geq 1$.

We will denote the disentanglement map by $f_{\mathbb{C}}$. The restriction of this map to the real part of the source will be denoted $f_{\mathbb{R}}$ and will be called a real perturbation. By abusing notation we will speak of $\mathrm{Dis}_k(f_{\mathbb{C}})$ the kth complex disentanglement and $\mathrm{Dis}_k(f_{\mathbb{R}})$ a kth real disentanglement. We wish to compare the topology of real and complex disentanglements.

In [12] and [17] when $n = 1$ a real perturbation is called good if the topology of the complex map can be seen in the topology of the real map. That is, if $\mathrm{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \mathrm{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$. In the case of general n the image of the complex map has homology in dimensions $n+1$ and 2 and so the following would seem a reasonable generalisation.

Definition 5.1 *If*

(i) $\mathrm{rank} H_{n+1}(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \mathrm{rank} H_{n+1}(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$ *and*

(ii) $\mathrm{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \mathrm{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$,

then $f_{\mathbb{R}}$ *is called a good real perturbation.*

Of course when $n = 1$ the two conditions in the definition are the same.

One of the aims of an investigation of this type is to show that if $f_{\mathbb{R}}$ is a good real perturbation then the real and complex disentanglements are homotopically equivalent. This is done for the case $n = 1$ in Theorem 5.5 but in general we use the weaker assumption that there is a $\mathbb{Z}$-homology equivalence between the real and complex disentanglements, see Theorem 5.4. First we need the following lemma.

Lemma 5.2 *The natural map* $H_0^{alt}(D^2(f_{\mathbb{R}}); \mathbb{Z}) \to H_0^{alt}(D^2(f_{\mathbb{C}}); \mathbb{Z})$ *is surjective. Generators of* $H_0^{alt}(D^2(f_{\mathbb{R}}); \mathbb{Z})$ *map to zero or generators of* $H_0^{alt}(D^2(f_{\mathbb{C}}); \mathbb{Z})$.

Proof. The map $f_{\mathbb{C}}$ is in effect a map from $U_1 \amalg \cdots \amalg U_m$ into $\mathbb{C}^{3n}$, where m is the number of elements of S. From the proof of Theorem 3.1 we know that $H_0^{alt}(D^2(f_{\mathbb{C}}); \mathbb{Z} \cong \mathbb{Z}^{m(m-1)/2}$. Furthermore, this group is generated by

$$H_0^{alt}(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \cup (U_j \times U_i)); \mathbb{Z}) \cong \mathbb{Z}$$

for each distinct U_i and U_j. This shows that $D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ is path connected. The groups $H_0^{alt}(D^2(f_{\mathbb{C}}) \cap (U_i \times U_i); \mathbb{Z})$ are all trivial so every component of $D^2(f_{\mathbb{C}}) \cap (U_i \times U_i)$ must intersect the diagonal.

Suppose that $(x_1, x_2) \in D^2(f_{\mathbb{R}})$ is contained in some path connected component X such that $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2) \neq 0$ is a generator of $H_0^{alt}(D^2(f_{\mathbb{R}}); \mathbb{Z})$.

If x_1 and x_2 are in the same U_i then $H_0^{alt}(D^2(f_{\mathbb{R}}) \cap \mathrm{Orbit}(X); \mathbb{Z})$ is $\mathbb{Z}_2$ or $\mathbb{Z}$ depending on whether $\mathrm{Orbit}(X)$ is connected or disconnected, see Lemma 2.6 of [4].

In both these situations this generator maps to zero as $H_0^{alt}(D^2(f_{\mathbb{C}}) \cap (U_i \times U_i); \mathbb{Z})$ is zero.

If x_1 and x_2 are in distinct U_i and U_j then $(x_1, x_2) \in D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ is connected to all points of $D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ and hence $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2)$ is a generator of $H_0^{alt}(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \cup (U_j \times U_i)); \mathbb{Z})$.

Now to prove the surjectivity statement. Suppose that B_ϵ is a Milnor ball and D_δ is a Milnor disc for the disentanglement map f_t. Then there exists a Milnor ball $B_{\epsilon'}$ and Milnor disc $D_{\delta'}$ with $0 < \epsilon' \leq \epsilon$ and $0 < \delta' \leq \delta$. The ball B_ϵ and disc $D_{\delta'}$ (i.e. a combination of two sets of data) will function as the Milnor ball and Milnor disc for the disentanglement map f_t with $t \in D_{\delta'}$. What we are effectively ensuring is that any change in topology occurs within $B_{\epsilon'}$. In fact using the Second Thom-Mather Isotopy Lemma in a similar way to [10] the map

$$f'_t := F|F^{-1}(B_\epsilon \times \{t\} - B_{\epsilon'} \times \{t\})$$

is topologically right-left equivalent to the map

$$f' := F|F^{-1}(B_\epsilon \times \{0\} - B_{\epsilon'} \times \{0\}),$$

where F is the map defined in Section 2.4. The map f is finitely $\mathcal{A}$-determined and so for every pair U_i, U_j with $i \neq j$ the images of $f|U_i$ and $f|U_j$ are transverse outside the origin and the same is true for the resulting components in f'. As f' is just the complexification of a real map the real parts of the pairs of components must be transverse in the image and under perturbation, i.e. in the image of the map f'_t, they must also be transverse. Hence for every pair U_i and U_j there exists a pair of real points with the same image in f_t. This pair will also generate $H_0^{alt}(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \times (U_j \times U_i)); \mathbb{Z})$ and so we get the surjectivity stated. □

Remark 5.3 *This result generalises Lemma 2.4 of [12]. However the proof given there is incorrect as it relies on the rational cohomology group $H^1(D^2(f_{\mathbb{C}}) \cap (U_i \times U_j); \mathbb{Q})$ being non-trivial. This is not the case for example if $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ arises from four planes forming an ordinary quadruple point, then the double point space for the disentanglement map is contractible to a finite set of points.*

The next theorem allows us to convert $\mathbb{Z}$-homology equivalences into homotopy equivalences.

Theorem 5.4 *Suppose that $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ is a finitely $\mathcal{A}$-determined multi-germ. If the natural inclusion $\mathrm{Dis}_1(f_{\mathbb{R}}) \hookrightarrow \mathrm{Dis}_1(f_{\mathbb{C}})$ is a $\mathbb{Z}$-homology equivalence, then it is a homotopy equivalence and all the triple points of f_t are real.*

Proof. Consider the pair of maps $f_{\mathbb{R}}$ and $f_{\mathbb{C}}$. We have an image computing spectral sequence,

$$E^1_{p,q}(f_{\mathbb{C}}, f_{\mathbb{R}}) = H_q^{alt}(D^{p+1}(f_{\mathbb{C}}), D^{p+1}(f_{\mathbb{R}}); \mathbb{Z}) \implies H_*(\mathrm{Dis}_1(f_{\mathbb{C}}), \mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}).$$

The only possible non-zero groups in E^1 are $E^1_{1,q}$ for $1 \leq q \leq \dim_{\mathbb{C}} D^2(f_{\mathbb{C}})$, ($E^1_{1,0} = H_0^{alt}(D^2(f_{\mathbb{C}}), D^2(f_{\mathbb{R}}); \mathbb{Z}) = 0$ by Lemma 5.2), and $E^1_{2,0}$, (since $D^3(f_{\mathbb{C}})$ and $D^3(f_{\mathbb{R}})$ are finite).

The differentials must all be zero and the sequence collapses at E^1. Any non-zero $E^1_{p,q}$ will contribute to $H_{p+q}(\mathrm{Dis}_1(f_{\mathbb{C}}), \mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z})$. But as $\mathrm{Dis}_1(f_{\mathbb{R}}) \to \mathrm{Dis}_1(f_{\mathbb{C}})$ is a $\mathbb{Z}$-homology equivalence all relative homology groups are zero, hence all entries of $E^1_{p,q}(f_{\mathbb{C}}, f_{\mathbb{R}})$ are zero.

This implies that the spectral sequence morphism $E^1_{p,q}(f_{\mathbb{R}}) \to E^1_{p,q}(f_{\mathbb{C}})$ is an isomorphism. In particular the complex $E^1_{*,0}(f_{\mathbb{R}})$ has the same form as $E^1_{*,0}(f_{\mathbb{C}})$ and it is this form, via Proposition 4.21 of [4], that implies the image of $f_{\mathbb{C}}$ is simply connected, i.e. the image of $f_{\mathbb{R}}$ is simply connected as well. (The β condition in Proposition 4.21 of [4] is that the map should be complex analytic but the proof works just as well for real analytic maps.)

Thus $\mathrm{Dis}_1(f_{\mathbb{R}}) \to \mathrm{Dis}_1(f_{\mathbb{C}})$ is a $\mathbb{Z}$-homology equivalence between simply connected CW-complexes and so is a homotopy equivalence by a theorem of Whitehead, see [18] p.182 and p.220.

The fact that $H_0^{alt}(D^3(f_{\mathbb{R}});\mathbb{Z}) \to H_0^{alt}(D^3(f_{\mathbb{C}});\mathbb{Z})$ is an isomorphism implies that all complex triple points are real. $\square$

Theorem 5.5 *Suppose $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ is a finitely $\mathcal{A}$-determined multi-germ such that* $\mathrm{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z}) = \mathrm{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$. *Then* $\mathrm{Dis}_1(f_{\mathbb{R}})$ *and* $\mathrm{Dis}_1(f_{\mathbb{C}})$ *are homotopically equivalent.*

Proof. First, by Theorem 1.1 of [17], the natural map $H_2(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z}) \to H_2(\mathrm{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$ is an isomorphism. Second, a consequence of Theorem 2.3 of [12] is that $H^i(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Q}) \to H^i(\mathrm{Dis}_1(f_{\mathbb{C}});\mathbb{Q})$ is an isomorphism for all i. Using these facts and the fact that $\mathrm{Dis}_1(f_{\mathbb{C}})$ is homotopically equivalent to a wedge of 2-spheres we can deduce that the only possible obstruction to an isomorphism on integer homology is if $H_1(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z})$ has non-trivial torsion.

There are two possible ways this could occur, we shall show that both are impossible in our situation.

If $D^2(f_{\mathbb{R}})$ contains a circle C upon which the natural S_2 action is the antipodal action, then $H_0^{alt}(D^2(f_{\mathbb{R}});\mathbb{Z})$ contains the subgroup $H_0^{alt}(C;\mathbb{Z}) \cong \mathbb{Z}_2$, (by Lemma 2.6 of [4]), and vice versa. If this torsion group survives in $E^2(f_{\mathbb{R}})$ then it appears as a summand in $H_1(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z})$.

Thus suppose such a cirle C exists in $D^2(f_{\mathbb{R}})$. Then C projects to a curve C' in the source of $f_{\mathbb{R}}$, the nodes of which correspond to triple points of $f_{\mathbb{R}}$. By performing homotopies on $f_{\mathbb{R}}$ we can remove any other points arising from the double point space that lie within the compact region enclosed by C'. That is we push them to outside the enclosed region.

There will exist a node such that on one side of it the curve forms a loop with no other nodes on it. That is the loop will be equivalent to the interval $[0, 1]$ mapped so that the points $\{\frac{1}{4}\}$ and $\{\frac{3}{4}\}$ form the node. Call this space α (its shape resembles an alpha). To one side of the plane containing the node the surface will look like the product of an interval I with α. By a smooth homotopy one can contract the loop $\{\frac{1}{2}\} \times \alpha$; this will modify $I \times \alpha$ to a space with two cross caps. The plane containing the node can then be pushed through the nearest cross cap thus removing the node. See Figure 1. This can be done for all nodes so that the projection of C for the new map gives a circle C' in the source with no nodes and no points of the double point space touching it or contained within its interior.

Once this is done the disc in the source with C' as boundary is mapped into $\mathbb{R}^3$ with C' mapping two-to-one. The image of this disc is in effect a copy of $\mathbb{RP}^2$ embedded in $\mathbb{R}^3$. Such a situation is impossible to achieve and hence there are no loops like C in $D^2(f_{\mathbb{R}})$.

Since $H_1(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z}) \cong E^2_{1,0}(f_{\mathbb{R}})$ the next possible way to get torsion is if the group

$$E^2_{1,0} \cong \mathrm{Ker}(d : E^1_{1,0} \to E^1_{0,0})/\mathrm{Im}(d : E^1_{2,0} \to E^1_{1,0})$$

contains some.

The differential $d : E^1_{2,0} \to E^1_{1,0}$ is induced from ε_{3*}, see Theorem 2.10. Suppose that (a, b, c) is a triple point then it is easy to calculate that

$$\varepsilon_{3*}(\mathrm{Alt}_{\mathbb{Z}}(a, b, c)) = \mathrm{Alt}_{\mathbb{Z}}(a, b) + \mathrm{Alt}_{\mathbb{Z}}(b, c) + \mathrm{Alt}_{\mathbb{Z}}(c, a).$$

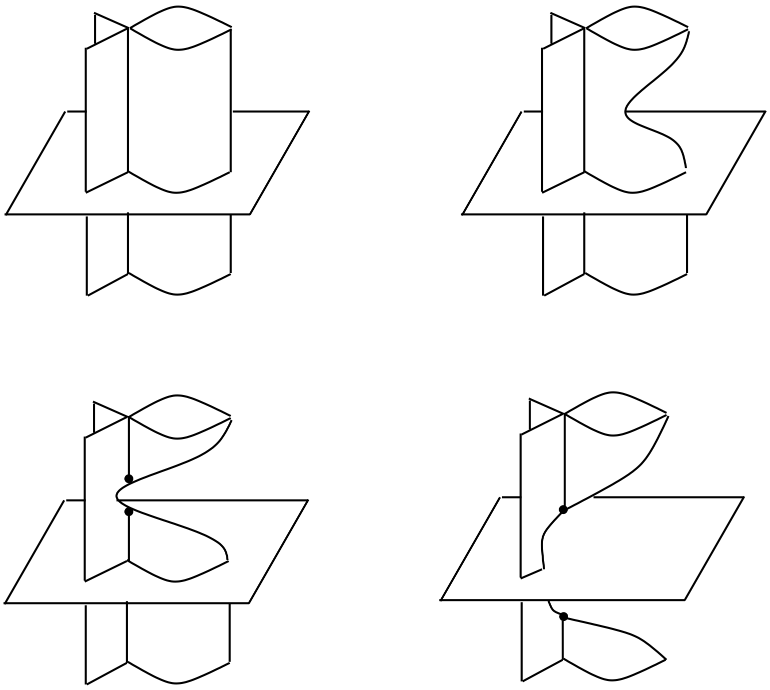

Figure 1: Removing nodes on C'.

So, if for example (a,b), (b,c) and (c,a) are all contained in the same component of an ordinary loop in $D^2(f_{\mathbb{R}})$ then $\varepsilon_{3*}(\mathrm{Alt}_{\mathbb{Z}}(a,b,c)) = 3\,\mathrm{Alt}_{\mathbb{Z}}(a,b)$, which could lead to torsion.

The double point space $D^2(f_{\mathbb{R}})$ contains components homeomorphic to the following forms: intervals with a point in the diagonal, circles which intersect the diagonal in two points and pairs of loops such that the pair is invariant under S_2.

The first two situations do not contribute anything to $H_0^{alt}(D^2(f_{\mathbb{R}});\mathbb{Z})$ as for these any point (x_1,x_2) is homologous to one in the diagonal, (x,x) say, and so $\mathrm{Alt}_{\mathbb{Z}}(x_1,x_2) \equiv \mathrm{Alt}_{\mathbb{Z}}(x,x) = 0$.

Thus, only pairs of loops could cause difficulties. Let C be such a loop and (x_1,x_2) be a point of C. The generator $\mathrm{Alt}_{\mathbb{Z}}(x_1,x_2)$ of $H_0^{alt}(D^2(f_{\mathbb{R}})\cap \mathrm{Orbit}(C)) \cong \mathbb{Z}$ is also a generator of some $\mathbb{Z}$ in $H_0^{alt}(D^2(f_{\mathbb{C}});\mathbb{Z})$, see Lemma 5.2.

We have the natural commutative diagram,

$$\begin{array}{ccccccc} E^1_{0,1}(f_{\mathbb{R}}) & \cong & \mathbb{Z}^{\mu} & \leftarrow & \mathbb{Z}^T & \cong & E^1_{2,0}(f_{\mathbb{R}}) \\ & & \downarrow & & \downarrow & & \\ E^1_{0,1}(f_{\mathbb{C}}) & \cong & \mathbb{Z}^{m(m-1)/2} & \leftarrow & \mathbb{Z}^T & \cong & E^1_{2,0}(f_{\mathbb{C}}). \end{array}$$

The first vertical arrow is surjective and takes generators of $H_0^{alt}(D^2(f_{\mathbb{R}});\mathbb{Z})$ to generators of $H_0^{alt}(D^2(f_{\mathbb{C}});\mathbb{Z})$, as just described and the second is an isomorphism as by Corollary 2.5 of [12] all triple points are real. Thus, if (a,b,c) is a triple point in $D^3(f_{\mathbb{R}})$ so that $\varepsilon_{3*}\,\mathrm{Alt}_{\mathbb{Z}}(a,b,c) \equiv r\,\mathrm{Alt}_{\mathbb{Z}}(x_1,x_2)$ with $r > 1$ in the top row, then a similar situation takes place in the bottom row leading to torsion in $E^2_{1,0}(f_{\mathbb{C}}) \cong H_1(\mathrm{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$. This is a contradiction.

Thus $H_1(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z}) = 0$ and so there is an isomorphism $H_i(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z}) \to H_i(\mathrm{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$ for all i. By Theorem 5.4 this leads to a homotopy equivalence. $\square$

Compare the following theorem with Theorem 3.6 of [8] where the homotopy conclusion is true when the image of the map is a hypersurface.

Theorem 5.6 *Suppose that $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a corank 1 finitely $\mathcal{A}$-determined map-germ. If the natural map $D^k(f_\mathbb{R}) \to D^k(f_\mathbb{C})$ is an S_k-homotopy equivalence for all $k \geq 2$ then the inclusion $\mathrm{Dis}_r(f_\mathbb{C}) \to \mathrm{Dis}_r(f_\mathbb{C})$ is a homotopy equivalence for all $r \geq 1$.*

Proof. The natural maps $H_i^{alt}(D^k(f_\mathbb{R}); \mathbb{Z}) \to H_i^{alt}(D^k(f_\mathbb{C}); \mathbb{Z})$ are isomorphisms for all $k \geq 1$ and thus the spectral sequence morphsism $E^1_{p,q}(f_\mathbb{R}) \to E^1_{p,q}(f_\mathbb{C})$ is an isomorphism for all p and q. This gives a $\mathbb{Z}$-homology equivalence between $\mathrm{Dis}_1(f_\mathbb{R})$ and $\mathrm{Dis}_1(f_\mathbb{C})$ and so by Theorem 5.4 the map is a homotopy equivalence.

The second disentanglements are the images of $f_\mathbb{R}|D^2_1(f_\mathbb{R}) : D^2_1(f_\mathbb{R}) \to M_2(f_\mathbb{R})$ and $f_\mathbb{C}|D^2_1(f_\mathbb{C}) : D^2_1(f_\mathbb{C}) \to M_2(f_\mathbb{C})$. The maps $H_i(D^2_1(f_\mathbb{R}); \mathbb{Z}) \to H_i(D^2_1(f_\mathbb{C}); \mathbb{Z})$ are isomorphisms as the spectral sequences for $g_\mathbb{R} : D^2(f_\mathbb{R}) \to D^2_1(f_\mathbb{R})$ and $g_\mathbb{C} : D^2(f_\mathbb{C}) \to D^2_1(f_\mathbb{C})$ are isomorphic (their multiple points spaces are S_k-homotopy equivalent). These then provide the isomorphisms between the spectral sequences for $f_\mathbb{R}|D^2_1(f_\mathbb{R})$ and $f_\mathbb{C}|D^2_1(f_\mathbb{C})$. Thus $M_2(f_\mathbb{R}) = \mathrm{Dis}_2(f_\mathbb{R})$ and $M_2(f_\mathbb{C}) = \mathrm{Dis}_2(f_\mathbb{C})$ are $\mathbb{Z}$-homology equivalent.

In a similar way one can prove that the spectral sequences for calculating $D^2(f_\mathbb{R})/S_2$ and $D^2(f_\mathbb{C})/S_2$ are isomorphic. If $n = 1$ then both $D^2(f_\mathbb{R})/S_2$ and $D^2(f_\mathbb{C})/S_2$ are homotopically equivalent to connected 1-dimensional CW-complexes and as they are $\mathbb{Z}$-homology equivalent they must be homotopy equivalent. If $n > 1$ then $D^2(f_\mathbb{R}) \simeq D^2(f_\mathbb{C}) \simeq \vee S^n$ (as $D^2(f_\mathbb{C})$ is the Milnor fibre of an isolated complete intersection singularity) is simply connected and as $H_0^{alt}(D^2(f_\mathbb{R}); \mathbb{Z}) = 0$ we deduce that $D^2(f_\mathbb{R})/S_2$ is simply connected. This is essentially done the same way as the proof of Theorem 4.1. So $D^2(f_\mathbb{R})/S_2$ and $D^2(f_\mathbb{C})/S_2$ are simply connected and $\mathbb{Z}$-homology equivalent CW-complexes and thus they are homotopically equivalent by Whitehead's theorem.

Now, just as $\mathrm{Dis}_2(f_\mathbb{C})$ is homotopically equivalent to $D^2(f_\mathbb{C})/S_2$ union some hexagons so is $\mathrm{Dis}_2(f_\mathbb{R})$ homotopically equivalent to $D^2(f_\mathbb{R})/S_2$ union some hexagons. See the proof of Theorem 4.4. In fact, $\mathrm{Dis}_2(f_\mathbb{C})$ is homotopically equivalent to $D^2(f_\mathbb{C})/S_2$ wedged with $2T(f_\mathbb{C})$ circles; each hexagon contracts to give these loops. Similarly for $\mathrm{Dis}_2(f_\mathbb{R})$. The homotopy equivalence from $D^2(f_\mathbb{R})/S_2$ to $D^2(f_\mathbb{C})/S_2$ can then be extended to one from $\mathrm{Dis}_2(f_\mathbb{R})$ to $\mathrm{Dis}_2(f_\mathbb{C})$.

That $\mathrm{Dis}_3(f_\mathbb{R})$ and $\mathrm{Dis}_3(f_\mathbb{C})$ are homotopically equivalent follows from the fact that $D^3(f_\mathbb{R})$ and $D^3(f_\mathbb{C})$ are S_3-homotopically equivalent and $M_3(f_\mathbb{C}) = D^3(f_\mathbb{C})/S_3$, etc. □

5.2 A family with excellent real perturbations

Definition 5.7 *As in [8] we call a real perturbation excellent if the natural map*

$$H_i(\mathrm{Dis}_r(f_\mathbb{R}); \mathbb{Z}) \to H_i(\mathrm{Dis}_r(f_\mathbb{C}); \mathbb{Z})$$

is an isomorphism for all i and r.

In [12] it is shown that every member of the H_k family given by $f(x, y) = (x, y^3, xy + y^{3k-1})$, $k \geq 1$, has a good real perturbation.

This family can be generalised to $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$ by taking

$$\begin{aligned}(x_1, \ldots, x_{2n-1}, y) \quad \mapsto \quad & \Big(x_1, \ldots, x_{2n-1}, y^3 + y \sum_{j=1}^{n-1} x^2_{n+j-1}, x_{2n-1}y + y^{3k-1}, \\ & x_1 y + x_n y^2, \ldots, x_i y + x_{i+n-1} y^2, \ldots, x_{n-1} y + x_{2n-2} y^2\Big).\end{aligned}$$

By taking $n = 1$ we get the H_k family. We will show that the general family has more than a good real perturbation, in fact the corresponding rth disentanglements are homotopy equivalent for all r.

Theorem 5.8 *The map $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$ is finitely $\mathcal{A}$-determined.*

Proof. Since the map is corank 1 if we show that $D^2(f)$ and $D^3(f)$ (defined by the adapted Vandermonde method of [11]) are isolated complete intersection singularities and $D^4(f)$ is empty, then by Theorem 2.14 of [11] it follows that f is finitely determined.

If we let $f(x, y) = (x, f_1(x, y), \ldots, f_{n+1}(x, y))$ then according to [11] $D^2(f)$ is generated in $\mathbb{C}^{2n-1} \times \mathbb{C}^2$ by

$$\frac{f_j(x, y_1) - f_j(x, y_2)}{y_1 - y_2}, \text{ for } j = 1, \ldots, n+1.$$

Thus it is generated by $y_1^2 + y_1 y_2 + y_2^2 + \sum_{j=1}^{n-1} x_{n+j-1}^2$, $x_{2n-1} + O(3k-2)$, (i.e. terms of degree of $3k-2$ and above), and $x_i + x_{i+n-1}(y_1 + y_2)$ for $1 \leq i \leq n-1$ and is equivalent to a quadratic isolated hypersurface singularity.

The set $D^3(f)$ is generated by

$$\frac{\begin{vmatrix} 1 & y_1 & f_j(x, y_1) \\ 1 & y_2 & f_j(x, y_2) \\ 1 & y_3 & f_j(x, y_3) \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}} \quad \text{and} \quad \frac{\begin{vmatrix} 1 & f_j(x, y_1) & y_1^2 \\ 1 & f_j(x, y_2) & y_2^2 \\ 1 & f_j(x, y_3) & y_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}$$

for $j = 1, \ldots, n+1$. These give the same generators as in the $n = 1$ case plus all x_i for $i = 1, \ldots, 2n-2$. Thus the $D^3(f)$ singularity is equivalent to the singularity arising in the $n = 1$ case and hence is an isolated complete intersection singularity.

The set $D^4(f)$ is empty because of the presence of the y^3 term in f_1. $\square$

Proposition 5.9 *For f above we have*

(i) $\mathrm{Dis}_1(f)$ *is homotopically equivalent to the wedge of an $(n+1)$-sphere and $k-1$ 2-spheres;*

(ii) $\mathrm{Dis}_2(f)$ *is homotopically equivalent to a wedge of $2(k-1)$ circles.*

Proof. For part (i) we know from Theorem 3.4 that $\mathrm{Dis}_1(f)$ is homotopically equivalent to a wedge of $(n+1)$-spheres and 2-spheres. The number of $(n+1)$-spheres is given by $\mathrm{rank} H_n^{alt}(D^2(f_t); \mathbb{Z})$ and since $D^2(f)$ is given by an S_2-invariant quadratic singularity this rank must be equal to 1. By Theorem 3.1 the number of 2-spheres is equal to $T(f_t)$, the number of triple points in the image of f_t. By the proof of Theorem 5.8 $D^3(f)$ is equivalent to the multiple point space in the case $n = 1$ and by [12] the degree of this is $6(k-1)$ and so the number of triple points of f_t is equal to $k-1$, as each triple point gives rise to an S_3-orbit of 6 points.

To prove part (ii) we use Theorem 4.4 to show that $\mathrm{Dis}_2(f)$ is homotopically equivalent to a wedge of n-spheres and $2T(f_t)$ circles. The number of n-spheres is equal to the rank of the symmetric part of $H_n(D^2(f_t); \mathbb{Z})$ and hence is zero. $\square$

Proposition 5.10 *The perturbation of f given by*

$$(x_1, \ldots, x_{2n-1}, y^3 + y \sum_{j=1}^{n-1} x_{n+j-1}^2 - ty, x_{2n-1} y + y^2 \left(\prod_{i=1}^{k-1} (y^3 - ty - c_i) \right),$$
$$x_1 y + x_n y^2, \ldots, x_i y + x_{i+n-1} y^2, \ldots, x_{n-1} y + x_{2n-2} y^2),$$

where $t > 0$ and c_i are real numbers with $|c_i| < \frac{2}{3} t^{3/2}$ and $c_i \neq c_j$ for $i \neq j$, is a good real perturbation. Furthermore, the natural map $\mathrm{Dis}_r(f_\mathbb{R}) \hookrightarrow \mathrm{Dis}_r(f_\mathbb{C})$ is a homotopy equivalence for all $r \geq 1$.

Proof. The first statement obviously follows from the second statement.

To prove the second we note that by the material in section 3 of [12] in the case $n = 1$ the map $D^3(f_{\mathbb{R}}) \to D^3(f_{\mathbb{C}})$ is a homeomorphism respecting the S_3 action. Since $D^3(f_{\mathbb{C}})$ is equivalent to this case we have an S_3-homotopy equivalence. The map $D^2(f_{\mathbb{R}}) \to D^2(f_{\mathbb{C}})$ is an S_2-equivariant homotopy equivalence, this follows from the description of $D^2(f)$. Thus the proposition is proved by Theorem 5.6. $\square$

6 Calculating the number of triples via weights and degrees

If we suppose that $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$ is corank 1 then, as we have seen, defining equations for the multiple point spaces are known. In the quasihomogeneous case we can use variable weights and definining equation degrees to calculate the Milnor numbers of the multiple point spaces and their restrictions to the diagonal.

Theorem 6.1 *Suppose that $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a quasihomogeneous corank 1 finitely $\mathcal{A}$-determined map-germ. Then, f is $\mathcal{A}$-equivalent to a map of the form*

$$(x_1, \dots, x_{2n-1}, y) \mapsto (x_1, \dots, x_{2n-1}, f_1(x, y), \dots, f_{n+1}(x, y)).$$

Then, the weight of the variables and the degrees of the functions $f_i(x, y)$, $i = 1, \dots, n+1$, can be used to calculate the number of triple points in the disentanglement:

$$T(f_t) = \frac{\prod_{j=1}^{n+1} (degree(f_j) - 2weight(y))(degree(f_j) - weight(y))}{6weight(y)^3 \prod_{j=1}^{2n-1} weight(x_j)}.$$

Proof. One uses the generators for $D^3(f)$ given in Theorem 5.8. The generators on the left in the proof have degree equal to the degree of f_j minus twice the weight of y. Those on the right have degree equal to the degree of f_j minus the weight of y. These generators are used in the space given by variables $x_1, \dots, x_{2n-1}$ and y_1, y_2, y_3.

As $D^3(f)$ is zero-dimensional the number of triple points under deformation (i.e. those in the disentanglement) is equal to $\frac{1}{6}$ of the degree of $D^3(f)$. This degree is equal to the product of the degrees of the generators divided by the product of the weights of the variables. $\square$

Formulae for the number of $(n+1)$-spheres and n-spheres in the first and second disentanglements using weights and degrees can be also be provided. See [2] and [15] for first disentanglements when $n \leq 2$. In general one calculates the Milnor number of the singularity $D^2(f)$ and the one for its restriction to the diagonal H using weights and degrees, and then uses these in the formulae:

Lemma 6.2 *Suppose $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$ is a corank 1 finitely $\mathcal{A}$-determined map-germ. Then,*

(i) *the number of $(n+1)$-spheres arising from the double point space of f_t in $\mathrm{Dis}_1(f)$ is equal to $\frac{1}{2}(\mu(D^2(f)) + \mu(D^2(f)|H))$;*

(ii) *the number of n-spheres arising from the double point space of f_t in $\mathrm{Dis}_2(f)$ is equal to $\frac{1}{2}(\mu(D^2(f)) - \mu(D^2(f)|H))$.*

Proof. For part (i) see [6]. Part (ii) is just from Theorem 4.4. $\square$

A formula for calculating Milnor numbers from weights and degrees is given in [3]: Suppose X is an r-dimensional isolated complete intersection singularity defined by quasi-homogeneous functions with degrees $d_1, \ldots, d_k$ and the weights of the variables are given by $w_1, \ldots, w_m$. Then,

$$\mu(X) = (-1)^{r+1} + \frac{d_1 \ldots d_k}{w_1 \ldots w_m} \sum_{\lambda=0}^{r} (-1)^{r-\lambda} D_\lambda W_{r-\lambda},$$

where $W_0 = 1$,

$$W_\lambda = \sum_{1 \leq \nu_1 < \cdots < \nu_\lambda \leq m} w_1 \ldots w_{\nu_\lambda}, \ \lambda \geq 1,$$

and $D_\lambda = \sum_{|\nu|=\lambda} d_1^{\nu_1} \ldots d_k^{\nu_k}$.

For the singularity $D^2(f)$ we can use as coordinates $x_1, \ldots, x_{2n-1}, y_1, y_2$ where $\mathrm{weight}(y_1) = \mathrm{weight}(y_2) = \mathrm{weight}(y)$, and the degrees of the defining equations are $\mathrm{degree}(f_i) - \mathrm{weight}(y)$ for $i = 1, \ldots, n+1$.

For $D^2(f)|H$ the variables are $x_1, \ldots, x_{2n-1}, y$ and degrees are $\mathrm{degree}(f_i) - \mathrm{weight}(y)$ for $i = 1, \ldots, n+1$.

From these it is possible to write down general formulae for the number of spheres in the disentanglements but these are complicated and inelegant. The case $n = 2$ given in [2] shows just how complicated the formula can get. It is probably better just to use the formulae for the individual multiple point spaces first.

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