Homotopy Type of Disentanglements of Multi-germs

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Abstract

For a complex analytic map f from n-space to p-space with $n < p$ and with an isolated instability at the origin, the disentanglement of f is a local stabilization of f that is analogous to the Milnor fibre for functions.

For mono-germs it is known that the disentanglement is a wedge of spheres of possibly varying dimensions. In this paper we give a condition that allows us to deduce that the same is true for a large class of multi-germs.

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1 Introduction

For a complex analytic map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ where S is a finite set of points and where the origin in $(\mathbb{C}^p, 0)$ has an isolated instability we can find a nearby stable map which we can view as a stabilization of f. We call the discriminant of this stabilization the disentanglement of f. It is analogous to the Milnor fibre of an isolated complete intersection singularity. It is well known that such a Milnor fibre is homotopically equivalent to a wedge of spheres. In the case of disentanglements it is known that for $n \geq p - 1$ that the disentanglement is homotopically a wedge of spheres of dimension $p - 1$, see [1, 7]. (These references give the statements and proofs for mono-germs but the multi-germ proof is practically the same.)

For mono-germs with $n < p - 1$ it was shown in [3] that the disentanglement is homotopically a wedge of spheres but the spheres can be of different dimensions. An outstanding problem which seems to be much harder is to describe the topology for the multi-germ case. In this paper we show in Theorem 2.2 that the integer homology of the disentanglement is free abelian and hence one could conjecture that in analogy with the other cases that the disentanglement is homotopically equivalent to a wedge of spheres. Using a rather nice trick we show in Theorem 3.1 that for a special but actually quite large class of maps this is indeed true.

2 Disentanglements

The details of the following definition of disentanglement can be found in [6]. The theorems there are stated for mono-germs but the extension to multi-germs is straightforward. Suppose S is a finite subset of $\mathbb{C}^n$ and that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$,

$n < p$, is a map-germ with an isolated instability at $0 \in (\mathbb{C}^p, 0)$ (equivalently, f is finitely $\mathcal{A}$-determined). Let $F : (\mathbb{C}^n \times \mathbb{C}^b, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}^b, 0)$ be a versal unfolding, so that F has the form $F(x,t) = (f_t(x), t)$. Let Σ be the bifurcation space in the unfolding parameter space $\mathbb{C}^b$, i.e., points such that the map $f_t : \mathbb{C}^n \to \mathbb{C}^p$ is unstable. For corank 1 maps and maps in the nice dimensions the set Σ is a proper subvariety of $\mathbb{C}^b$ and so $\mathbb{C}^b \backslash \Sigma$ is connected. In other cases we relax our condition and ask that f_t is topologically stable instead. In this case we also have that Σ is a proper subvariety of $\mathbb{C}^b$.

Given a Whitney stratification of the image of f in $\mathbb{C}^p$ there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, the real $(2p-1)$-sphere centred at 0 of radius ϵ is transverse to the strata of the image of f.

Consider the map $f_t | f_t^{-1}(B_\epsilon)$ where B_ϵ is the closed ball of radius ϵ centred at 0 with $\epsilon \leq \epsilon_0$ and $t \in \mathbb{C}^b \backslash \Sigma$. This stable map is called *the disentanglement map* of f and its image is the *disentanglement* of f, denoted $\mathrm{Dis}(f)$. This is independent of sufficiently small ϵ and t. To ease notation we will write $\widetilde{f}$ rather than $f_t | f_t^{-1}(B_\epsilon)$. As $f^{-1}(0)$ is a finite set we can assume that $\widetilde{f}$ has the form $\widetilde{f} : \coprod_{j=1}^{|S|} U_j \to \mathbb{C}^p$ where U_j is a contractible open set.

In the case of $n \geq p$ a similar construction can be made where instead of the image of f_t we use its discriminant.

The standard definition of the multiple point spaces of a map is the following.

Definition 2.1 *Suppose that $f : X \to Y$ is a continuous map of topological spaces. Then the k*th multiple point space of f, *denoted $D^k(f)$, is the set*

$$D^k(f) := \mathrm{closure}\{(x_1, \dots, x_k) \in X^k | f(x_1) = \dots = f(x_k), \text{ such that } x_i \neq x_j, i \neq j\}.$$

The group of permutations on k objects, denoted S_k, acts on $D^k(f)$ in the obvious way: permutation of copies of X in X^k.

Let $d(f) = \sup\{k \mid D^k(\widetilde{f}) \neq \emptyset\}$ and let $s(f)$ be the number of branches of f through the origin 0 in $(\mathbb{C}^p, 0)$, i.e., the cardinality of S.

We can now generalize Corollary 4.8 of [3] to the case of multi-germs. A version for rational cohomology for corank 1 multi-germs was given in [5].

Theorem 2.2 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, has an isolated instability at $0 \in (\mathbb{C}^p, 0)$. Then, $\widetilde{H}_*(\mathrm{Dis}(f); \mathbb{Z})$ is free abelian and has non-trivial groups possible only in dimensions $p - (p-n-1)k - 1$ for all $2 \leq k \leq d(f)$ and if $s(f) > d(f)$, then in $\widetilde{H}_{d(f)-1}(\mathrm{Dis}(f); \mathbb{Z})$.*

Furthermore, $H_0(\mathrm{Dis}(f); \mathbb{Z}) = \mathbb{Z}$, i.e., $\mathrm{Dis}(f)$ is connected.

If $p = n + 1$, then a simple corollary of this is that the only non-trivial reduced homology groups occur in dimension $p - 1$. This is known for mono-germs since the disentanglement is homotopically equivalent to a wedge of spheres, see [7].

Suppose X is a topological space with the homotopy type of a CW-complex such that S_k acts cellularly, that is, open cells go to open cells. Whitney stratified spaces can be triangulated so that they are CW-complexes. Furthermore, our multiple point spaces will be, generally speaking, Whitney stratified spaces such that S_k acts cellularly on the CW-complex, see [3].

Let $C_*(X; \mathbb{Z})$ denote the cellular chain complex of X.

Definition 2.3 *The alternating chain complex $C_*^{alt}(X; \mathbb{Z})$ is the subcomplex of $C_*(X; \mathbb{Z})$ given by*

$$C_n^{alt}(X; \mathbb{Z}) := \{c \in C_n(X; \mathbb{Z}) | \sigma c = \mathrm{sign}(\sigma) c \text{ for all } \sigma \in S_k\}.$$

The alternating homology of X is the homology of the complex $C_*^{alt}(X; \mathbb{Z})$ and is denoted $H_*^{alt}(X; \mathbb{Z})$.

We can use the alternating homology of multiple point spaces to calculate the homology of the image of a finite and proper map.

Theorem 2.4 *Let $f : X \to Y$ be a finite and proper subanalytic map and let Z be a (possibly empty) subanalytic subset of X such that $f|Z$ is also proper. Then, there exists a spectral sequence*

$$E_1^{r,q}(f, f|Z) = H_q^{alt}(D^{r+1}(f), D^{r+1}(f|Z); \mathbb{Z}) \Rightarrow H_*(f(X), f(Z); \mathbb{Z}),$$

where the differential is induced from the natural map $\varepsilon_{r+1,r} : D^{r+1}(f) \to D^r(f)$ given by $\varepsilon_{r+1,r}(x_1, x_2, \dots, x_r, x_{r+1}) = (x_1, x_2, \dots, x_r)$.

A proof is found in [4]. The precise details of the differential will not be required as our spectral sequences will be sparse. A spectral sequence for a single map rather than a pair also exists if we take $Z = \emptyset$.

Let U and W be open sets so that $F' : U \to W$ is a one-parameter unfolding of f of the form $F'(x, t) = (f_t(x), t)$ and f_t gives the disentanglement of f for $t \neq 0$.

Lemma 2.5 *The E_1 terms of the image computing spectral sequence of F' are*

$$E_1^{r,q}(F') \cong \begin{cases} \mathbb{Z}^{\binom{s(f)}{r+1}}, & \text{for } q = 0 \text{ and } 1 \le r+1 \le s(f), \\ 0, & \text{otherwise.} \end{cases}$$

The sequence collapses at E_2 and

$$E_\infty^{r,q}(F') \cong E_2^{r,q}(F') \cong \begin{cases} \mathbb{Z} & \text{for } (r,q) = (0,0), \\ 0 & \text{for } (r,q) \neq (0,0). \end{cases}$$

Proof. The proof is essentially the same as the proof of Lemma 3.3 of [5], only minor modifications need be made. □

We now prove an integer homology version of Lemma 3.4 of [5] that does not assume corank 1.

Lemma 2.6 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, has an isolated instability at 0 with a one-parameter unfolding F' and disentanglement map $\widetilde{f}$. As before let $d(f) = \sup\{k \mid D^k(\widetilde{f}) \neq \emptyset\}$ and $s(f)$ be the number of branches of f.*

Then,

$$E_1^{r,q}(F', \widetilde{f}) = \begin{cases} H^{alt}_{\dim_{\mathbb{C}} D^{r+1}(\widetilde{f})+1}(D^{r+1}(F'), D^{r+1}(\widetilde{f}); \mathbb{Z}), & \text{for } q = \dim_{\mathbb{C}} D^{r+1}(\widetilde{f}) + 1 \ge 0 \\ & \quad \text{and } r+1 \le d(f), \\ \mathbb{Z}^{\binom{s(f)}{r+1}}, & q = 0 \text{ and } r+1 > d(f), \\ 0, & \text{otherwise.} \end{cases}$$

Here we define $\dim \emptyset = -1$.

Proof. The proof is similar to the proof of Theorem 4.6 of [3]. The main difference is that $D^k(F')$ is no longer connected and in general neither is $D^k(\widetilde{f})$. This means we have to be careful about the bottom row of the spectral sequence, i.e., $E_1^{r,0}$.

By reasoning similar to the proof of Theorem 4.6 in [3] we have that

$$H_i^{alt}(D^k(F'), D^k(\widetilde{f}); \mathbb{Z}) = 0 \text{ for } i \le nk - p(k-1) = \dim_{\mathbb{C}} D^k(\widetilde{f}).$$

The main change in the proof is that the fibration g is a multi-germ fibration and we take the alternating homology of the fibres of this map.

As in Theorem 4.6 of [3] we have that $H_i^{alt}(D^k(\widetilde{f}); \mathbb{Z}) = 0$ for $i > \dim D^k(\widetilde{f})$ and is free abelian for $i = D^k(\widetilde{f})$.

Therefore for $r+1 \leq d(f)$ we have that

$$E_1^{r,q}(F', \widetilde{f}) = H^{alt}_{\dim_{\mathbb{C}} D^{r+1}(\widetilde{f})+1}(D^{r+1}(F'), D^{r+1}(\widetilde{f}); \mathbb{Z}) \text{ for } q = \dim_{\mathbb{C}} D^{r+1}(\widetilde{f}) + 1$$

and zero otherwise. If $r+1 > d(f)$, then $D^{r+1}(\widetilde{f})$ is empty by definition of $d(f)$ and so

$$E_1^{r,q}(F', \widetilde{f}) \cong E_1^{r,q}(F')$$

for all q. Therefore by Lemma 2.5 we have

$$E_1^{r,q}(F', \widetilde{f}) = \mathbb{Z}^{\binom{s(f)}{r+1}}$$

for $q = 0$ and zero otherwise. $\square$

We are now in a position to prove Theorem 2.2.

Proof (of Theorem 2.2). From Lemma 2.5 and Lemma 2.6 we can see that the bottom row of the image computing spectral sequence for $(F', \widetilde{f})$ is exact except possibly at $E_1^{d(f),0}$. Hence, $E_2^{r,0}(F', \widetilde{f}) = 0$ except possibly for $E^{d(f),0}(F', \widetilde{f})$ and this will be free abelian.

From Lemma 2.6 we can see that there are no other non-trivial differentials and so $E_2^{r,q}(F', \widetilde{f}) \cong E_1^{r,q}(F', \widetilde{f})$ for all r and all $q \neq 0$.

From the positions of the non-trivial groups in $E_2^{r,q}(F', \widetilde{f})$ we can see that the sequence collapses at E_2. Since there is no torsion there are no extension problems and we can deduce that

$$H_*(\operatorname{Im} F', \operatorname{Im} \widetilde{f}; \mathbb{Z}),$$

where Im denotes image, has non-trivial groups possible only in dimensions $p-(p-n-1)k$ for $2 \leq k \leq d(f)$ and in dimension $d(f)$ if $s(f) > d(f)$.

Since the image of F' has the homology of a point by Lemma 2.5 we deduce that $H_*(\operatorname{Dis}(f); Z) = H_*(\operatorname{Im} \widetilde{f}; \mathbb{Z})$ has the homology described in the statement of the theorem. $\square$

3 The homotopy type of the disentanglement

In this section we use a little trick to show that in a large number of cases the homotopy type of the disentanglement of a multi-germ with $n < p$ is homotopically a wedge of spheres of varying dimensions.

Theorem 3.1 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, is a multi-germ with an isolated instability at 0 and $s(f) \leq d(f)$. Then* $\operatorname{Dis}(f)$ *is homotopically equivalent to a wedge of spheres where the possible (real) dimensions are $p-(p-n-1)k-1$ for all $2 \leq k \leq d(f)$.*

To prove this theorem we shall construct a map, the image of which is homotopically equivalent to the disentanglement of f and such that its image computing spectral sequence is much simpler.

Lemma 3.2 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, is a multi-germ with an isolated instability at 0 and that $\widetilde{f} : \coprod_{j=1}^{s(f)} U_j \to \mathbb{C}^p$ is a stabilization giving the disentanglement.*

If $s(f) \leq d(f)$, then there exists $y \in \mathbb{C}^p$ such that $\widetilde{f}^{-1}(y) \cap U_j \neq \emptyset$ for all $j = 1, \ldots, s(f)$.

Proof. As before suppose that F' is one-parameter unfolding of f. By Lemma 2.6 we know that

$$H_0^{alt}(D^k(F'), D^k(\widetilde{f}); \mathbb{Z}) = 0$$

for all $k \leq d(f)$. In particular, $H_0^{alt}(D^{s(f)}(F'), D^{s(f)}(\widetilde{f}); \mathbb{Z}) = 0$. From Lemma 2.5 we know that $H_0^{alt}(D^{s(f)}(F'); \mathbb{Z}) = E_1^{s(f)-1,0}(F') = \mathbb{Z}$. This implies that the natural inclusion of $D^{s(f)}(\widetilde{f})$ into $D^{s(f)}(F')$ induces a surjection

$$H_0^{alt}(D^{s(f)}(\widetilde{f}); \mathbb{Z}) \to H_0^{alt}(D^{s(f)}(F'); \mathbb{Z})$$

(which in fact is an isomorphism if $\dim D^{s(f)}(\widetilde{f}) > 0$).

Thus there exists a point z in $D^{s(f)}(\widetilde{f})$ so that its orbit produces a generator of $H_0^{alt}(D^{s(f)}(\widetilde{f}); \mathbb{Z})$ which maps to the generator of $H_0^{alt}(D^{s(f)}(F'); \mathbb{Z})$. We can choose this z so that it is not in the image of the map $\varepsilon_{s(f)+1,s(f)}$ defined in Theorem 2.4. This is because this image will be of smaller dimension and hence will not disconnect $D^{s(f)}(\widetilde{f})$.

If $z = (z_1, z_2, \ldots, z_{s(f)}) \in \left(\coprod_{l=1}^{s(f)} U_l\right)^{s(f)}$, then as the (alternating) orbit of this point generates $H_0^{alt}(D^{s(f)}(F'); \mathbb{Z})$ we must have that each z_j and U_l can be uniquely paired. Since $z \in D^{s(f)}(\widetilde{f})$, there exists a $y \in \mathbb{C}^p$ such that $\widetilde{f}(z_j) = y$ for $1 \leq j \leq s(f)$. In other words we have shown that there exists $y \in \mathbb{C}^p$ such that $\widetilde{f}^{-1}(y) \cap U_j \neq \emptyset$ for all $j = 1, \ldots, s(f)$. $\square$

For a map $g : X \to Y$, let $M_k(g)$ be the image of the map from $D^k(g)$ to Y given by $(x_1, \ldots, x_k) \mapsto f(x_1)$.

Consider F' the one-parameter unfolding of f and let f_t be a family of maps so that $f_0 = f$ and f_t is a disentanglement map for $0 < t \leq 1$. The condition $s(f) \leq d(f)$ in the statement of the theorem implies via the preceding lemma that $M_{s(f)}(\widetilde{f})$ is non-empty as it contains y. (Note that the proof gives that $y \in M_{s(f)}(\widetilde{f}) \backslash M_{s(f)+1}(\widetilde{f})$.) This implies that $M_{s(f)}(f_t)$ must be non-empty for $t \neq 0$ also. Thus $M_{s(f)}(F')$ is at least one dimensional and as it is an analytic subspace of $\mathbb{C}^p \times \mathbb{C}^b$ it is path connected. Since this set must pass through the origin in $\mathbb{C}^p \times \mathbb{C}^b$ and from the path connectedness we know that there exists a path $\alpha : [0,1] \to M_{s(f)}(F')$ such that $\alpha(0) = 0$, $\alpha(t) \in M_{s(f)}(f_t) \backslash M_{s(f)+1}(f_t)$, and $\alpha(1) = y \in M_{s(f)}(\widetilde{f}) \backslash M_{s(f)+1}(\widetilde{f})$.

Let $Y = \mathrm{Dis}(f)$ and $Y^+ = (Y \coprod [0,1]) / \sim$ where $\sim$ is the identification of $y \in Y$ and $1 \in [0,1]$. Then obviously Y is homotopically equivalent to Y^+ as we have merely attached an interval to Y.

We shall attach intervals to each U_j and then identify their free ends to a single point. Let x_j be a point $\widetilde{f}^{-1}(y) \cap U_j$ that can be used in generating $H_0^{alt}(D^{s(f)}(\widetilde{f}); \mathbb{Z})$ as above and let $X_j^+ = (U_j \coprod [0,1]) / (x_j \sim 1)$. Next, let $X^+ = (\coprod X_j^+) / \sim$, where $\sim$ is the identification of all the origins of the intervals in the copies of $[0,1]$. This construction is such that X^+ is homotopically equivalent to a wedge of all the U_j.

Via straightforward inclusion we can consider X, U_j, etc. as subsets of X^+. Define $f^+ : X^+ \to Y^+$ by

$$\begin{cases} f^+ | \coprod U_j := \widetilde{f} \\ f^+ | X_j^+ \backslash U_j := \alpha(t) \text{ for all } j = 1, \ldots, s(f) \text{ and } t \in [0,1]. \end{cases}$$

It is easy to see that f^+ is a continuous map defined at all points of X^+ and its image is Y^+.

Lemma 3.3 *The set Y^+ is homotopically equivalent to a wedge of spheres with possible dimensions as in Theorem 3.1.*

Proof. Although the statement of Theorem 2.4 was for subanalytic maps the image computing spectral sequence exists for f^+ since $\widetilde{f}$ is a complex analytic map and the extra bits we add on to get f^+ are particularly simple, see [4] for details. That is, there exists a spectral sequence,

$$E_1^{r,q}(f^+) = H_q^{alt}(D^{r+1}(f^+);\mathbb{Z}) \implies H_*(Y^+;\mathbb{Z}).$$

We will show that the E^1 page of this sequence is particularly sparse and we shall compare it with the image computing spectral sequence for $\widetilde{f}$.

The obvious inclusions $h : \widetilde{Y} \to Y^+$ and $h_1 : \widetilde{X} \to X^+$ lead to inclusions $h_k : D^k(\widetilde{f}) \to D^k(f^+)$ for all $k \geq 2$.

As $\widetilde{f}$ and F are complex analytic, the multiple point spaces $D^k(\widetilde{f})$ are Whitney stratifiable and hence triangulable. This triangulation can be chosen such that we can construct $D^k(f^+)$ by attaching a number of 1-cells to 0-cells in $D^k(\widetilde{f})$ and then identifying the ends of the 1-cells to a single 0-cell which is invariant under the action of S_k. (Note that the action of S_k on the interior of the 1-cells is the same as the one on the 0-cells in $D^k(\widetilde{f})$, i.e., the ends of the 1-cells that are not fixed by the S_k-action.) For each k define V_k to be the closed cellular complex given by these 1-cells and 0-cells.

In summary we have $D^k(f^+) = D^k(\widetilde{f}) \cup V_k$ where $D^k(\widetilde{f}) \cap V_k$ is a finite collection of 0-cells. Call this intersection P_k. (Essentially these cells will arise from the orbit of k-tuples of the points z_j in the proof of Lemma 3.2.)

We can use an alternating homology version of the Mayer-Vietoris sequence to get a long exact sequence:

$$\begin{aligned} \cdots \to H_n^{alt}(D^k(\widetilde{f}) \cap V_k;\mathbb{Z}) &\to H_n^{alt}(D^k(\widetilde{f});\mathbb{Z}) \oplus H_n^{alt}(V_k;\mathbb{Z}) \\ &\to H_n^{alt}(D^k(f^+);\mathbb{Z}) \to H_{n-1}^{alt}(D^k(\widetilde{f}) \cap V_k;\mathbb{Z}) \to \dots \end{aligned}$$

The alternating homology of V_k is zero for all $k > 1$. This is because obviously no 1-cell can be a cycle and because all the 0-cells are homologous to the S_k-invariant 0-cell. This latter fact implies that the zeroth alternating homology group is zero by Lemma 2.6(iii) of [3].

Therefore the Mayer-Vietoris sequence gives the long exact sequence

$$\cdots \to H_n^{alt}(P_k;\mathbb{Z}) \to H_n^{alt}(D^k(\widetilde{f});\mathbb{Z}) \to H_n^{alt}(D^k(f^+);\mathbb{Z}) \to H_{n-1}^{alt}(P_k;\mathbb{Z}) \to \dots$$

The map $H_0^{alt}(P_k;\mathbb{Z}) \to H_0^{alt}(D^k(\widetilde{f});\mathbb{Z})$ is injective since the points in P_k were chosen to be generators of $H_0^{alt}(D^k(\widetilde{f});\mathbb{Z})$. (If $D^k(\widetilde{f})$ is not 0-dimensional, then it is in fact a bijection.)

Therefore we conclude that for $k \geq 2$.

$$H_n^{alt}(D^k(f^+);\mathbb{Z}) = \begin{cases} H_n^{alt}(D^k(\widetilde{f});\mathbb{Z}), & \text{for } n = \dim_{\mathbb{C}} D^k(\widetilde{f}), \\ 0, & \text{otherwise.} \end{cases}$$

If $k = 1$, then $D^k(f^+)$ is homotopically equivalent to the wedge of the U_j and so $H_*(D^k(f^+);\mathbb{Z})$ is just the ordinary homology of a point.

From this we deduce that

$$E_1^{r,q}(f^+) = \begin{cases} \mathbb{Z}^{\mu_r}, & \text{for } q = \dim_{\mathbb{C}} D^{r+1}(\widetilde{f}), r > 1, \text{ some } \mu_r \in \mathbb{N} \cup \{0\}, \\ \mathbb{Z} & \text{for } (r,q) = (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the image computing spectral sequence for f^+ degenerates at the first page, i.e., $E_m^{r,q}(f^+) = E_1^{r,q}(f^+)$ for all $m \in \mathbb{N}$. Since the groups are free abelian we can read off the homology of Y^+ from this sequence, noting that the possible dimensions with non-zero homology are those as in Theorem 3.1.

Then using the same reasoning as in Theorem 4.24 of [3] for the mono-germ case we can deduce that Y^+ is homotopically equivalent to a wedge of spheres. Note that, if Y has non-trivial homology in dimension 1, then this implies that Y is homotopically a wedge of circles and if $n < p-1$, then this is the only situation where Y is not simply connected. (If $n = p-1$, then we already know from [7] that the disentanglement is homotopically a wedge of spheres.) □

We now complete the proof of the theorem.

Proof (of Theorem 3.1). All we have to do now is show that Y and Y^+ have the same homotopy type. To do this we shall compare the spectral sequences for f^+ and $\widetilde{f}$.

We saw in the preceding lemma that the natural inclusion $h_k : D^k(\widetilde{f}) \to D^k(f^+)$ induces an isomorphism

$$H_n^{alt}(D^k(\widetilde{f}); \mathbb{Z}) \to H_n^{alt}(D^k(f^+); \mathbb{Z})$$

for all $n \geq 1$ and $k \geq 1$.

From the proof of Lemma 2.6 we can see that the bottom row of the spectral sequence for $\widetilde{f}$ may have non-zero terms, i.e., $E_1^{r,0}(\widetilde{f}) \cong E_1^{r,0}(F')$ for $r+1 \leq d(f)$ and $E_1^{r,0}(F')$ may be non-zero by Lemma 2.5. However, on the second page of the spectral sequence we have $E_1^{0,0}(\widetilde{f}) = \mathbb{Z}$ and only one other possible non-zero term at $E_1^{s(f),0}(\widetilde{f})$ (when $s(f) > d(f)$), see the proof of Theorem 2.2

Therefore the map $E_2^{r,q}(\widetilde{f}) \to E_2^{r,q}(f^+)$ is an isomorphism for all r and q. Since the sequence for f^+ degenerates at the first page and the one for $\widetilde{f}$ degenerates at the second page this implies that the natural map $h : Y \to Y^+$ induces an isomorphism on homology.

Now, by Proposition 4.21 of [3] we have that Y is simply connected. (The statement there is for a complex analytic map - a condition that a wayward Latex macro changed to a German double S.) By Lemma 3.3 and the comment at the end of its proof we have that Y^+ is simply connected or is homotopically a wedge of circles. Therefore, in the former case, by Whitehead's Theorem ([8] p 187) the map h induces an isomorphism on all homotopy groups. Since both spaces are triangulable this implies that they are homotopically equivalent, ([8] p 187). In the latter we note from the construction of the spectral sequence, see [2] or [4], that Y is homotopically equivalent to a space constructed from the contractible U_j by adding 1-cells that correspond to double points. Hence Y is also homotopically a wedge of circles. Now we merely note that this number of circles is the same as for Y^+ as h induces an isomorphism on homology. Hence Y and Y^+ are homotopically equivalent. □

Remarks 3.4 *(i). The class of germs in the theorem with $s(f) \leq d(f)$ is very large. Obviously, the condition trivially holds for mono-germs. Now consider a bi-germ f with branches f_1 and f_2. If any branch is not an immersion, say f_1, then $D^2(f_1) \neq \emptyset$. Now if the dimension of $D^2(f)$ is greater than 1, then as f_1 and f_2 will generally meet transversally (outside the origin), the intersection of the double points of f_1 and the image of f_2 will be non-empty. Thus we have a triple point. Again depending on dimension, this means that the disentanglement has a triple point. That is, $d(f) \geq 3$. Similar reasoning holds for any $s(f)$.*

(ii). We can construct maps with isolated instabilities such that $s(f) \geq d(f)$ and their difference is any arbitrary number. Consider s lines in the complex

plane all meeting at the origin. This is the image of a map with an isolated instability. The only type of singularity occurring in the disentanglement is an ordinary double point. Hence $d(f) = 2$ but $s(f) = s$ is arbitrary.

Note though that in this case the disentanglement is still homotopically equivalent to a wedge of spheres.

(iii). *If $s(f) > d(f)$, then Theorem 2.2 gives us the hope that the disentanglement is homotopically a wedge of spheres. As an example consider the ordinary quadruple surface point in three-space, that is, the image of four 2-planes into $\mathbb{C}^3$ such that the intersection of planes is pairwise transverse. It is easy to calculate that $s(f) = 4$ but that $d(f) = 3$ and that the disentanglement is homotopically equivalent to a single 2-sphere.*

(iv). *If we have a sufficiently generic finite complex analytic map from a complete intersection of dimension n to $\mathbb{C}^p$, $n < p$, then it is possible to show that the complex links of strata have free abelian integer homology, see Corollary 4.17 of [3]. It should be possible to use reasoning similar to the proof of Theorem 3.1 to show that a large class of the complex links are homotopically equivalent to a wedge of spheres of varying dimensions.*

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