Singularities in generic one-parameter complex analytic families of maps

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February 15, 2007

Abstract

We give a complete classification of complex analytic $\mathcal{A}_e$-codimension one multi-germs from one complex manifold to another in terms of mono-germs. (These are singularities one would expect in a generic one-parameter family of maps.) These multi-germs are described in terms of augmentations and concatenations of mono-germs, Morse singularities and a special bi-germ.

AMS Mathematics Subject Classification 2000 : 58K40, 58K65

1 Introduction

In the late 1960s John Mather greatly advanced the classification of singularities in a series of papers. Of particular significance is his classification of stable singularities in [15]. This is described as astonishing in [2], but it is perhaps underrated today because he achieved this tour-de-force in a single paper in the series, rather than producing a sequence of papers converging to the final classification.

Mather was classifying the singularities up to $\mathcal{A}$-equivalence, that is, up to diffeomorphism in source and target. This is an equivalence harder to work with and less amenable to study than say the coarser $\mathcal{K}$ or $\mathcal{R}$ equivalences and so there have not been many general classifications in the general style of Mather's stable classification. Instead classifications have been case-by-case and rather piecemeal: for specific pairs of dimensions, simple maps, low codimension or corank, or some other specific restrictions. See [2] for a large but far from complete list of classifications (and for an excellent introduction on how to efficiently compute such classifications). Explicit classifications under $\mathcal{A}$-equivalence can be found, for example, in [1], [8], [9], [14] and [16].

An obvious target for classification is the set of singularities appearing in generic one-parameter families of maps. The point being that in the space of maps one often wants to connect two stable ones by a path and hence one needs to know what singularities one would encounter if this path were generic.

For $\mathcal{A}$-classifications one can measure how degenerate the singularity is by using the $\mathcal{A}_e$-codimension. (See [19] for standard definitions within Singularity Theory.) The $\mathcal{A}_e$-codimension zero maps are precisely the stable maps and so the next natural target for classification are $\mathcal{A}_e$-codimension one maps. Recent progress in this area has been made in [3], [4], [6], [10] and [12]. In particular, in [6] Damon greatly illuminated the area by transferring the problem to an equivalent one involving what he called $\mathcal{K}_V$-equivalence.

The point is that in many situations the singularities occurring in generic one-parameter families are precisely the $\mathcal{A}_e$-codimension one germs.

At this stage we do not have a method to generate $\mathcal{A}_e$-codimension one germs in the simple and complete manner that Mather generated the stable germs. The aim of this paper is to describe such a method for germs with more than one branch, i.e. we classify multi-germs rather than mono-germs. Along the way a number of advances are made and material from a number of the papers listed previously is unified.

Perhaps the most significant advance is Theorem 3.3 which states that complex analytic $\mathcal{A}_e$-codimension one germs that are $\mathcal{K}$-equivalent are $\mathcal{A}$-equivalent. This is analogous to what happens in Mather's paper, it allows one to reduce to a $\mathcal{K}$-equivalence classification. It should be noted that Mather's theorem worked for real and complex maps, while the theorem here is, unfortunately, not true for real maps.

The outline of paper is as follows. Section 2 deals with augmentation of singularities. This process allows us to create new $\mathcal{A}_e$-codimension one germs from old ones and hence by reversing the process we can reduce our classification to simpler germs. Section 3 contains the proof that the '$\mathcal{K}$-equivalent implies $\mathcal{A}$-equivalent' theorem for $\mathcal{A}_e$-codimension one maps and the following section states a condition for a particular $\mathcal{K}$-class to have a codimension one map-germ associated with it.

Section 5 is again about creating new maps from old ones. Two types of *concatenation* are described. The definition given here of *monic concatenation* is effectively a combination of the definitions given in [4] and [12], thus unifying two separate cases. The idea is to take a germ and add a non-singular branch to the discriminant. The second type, *binary concatenation*, is the same as in [4]. It takes two germs, and then puts trivial extensions of their unfoldings in such a way as to be 'almost' transverse.

The notion of concatenation is, in some sense equivalent, to one that arises from product union in Damon's work. However, the notion here seems to be more useable, particularly in other contexts. First, it allows one to give more explicit descriptions of maps. Second, it gives a convenient way to describe different cases. For example, see [13] where the use of monic and binary allows one to describe easily the cases involved in dealing with the multiple point spaces of $\mathcal{A}_e$-codimension one maps.

In Section 6 commutativity and associativity relations between the three operations of augmentation and concatenation are found. These were initially proved through explicit diffeomorphisms in source and target in [12] in the case that the source dimension was less then the target dimension. Here the statements are for all pairs of dimensions and the proofs are greatly simplified through the use of Theorem 3.3.

The main classification theorem is Theorem 7.1. Here it is shown that complex $\mathcal{A}_e$-codimension 1 multi-germs can be constructed from $\mathcal{A}_e$-codimension one mono-germs using augmentation, monic concatenation and binary concatenation.

One improvement here is that the statement is for general n and p, where n is the source dimension of the map and p the target dimension. Previously the statements were for $n < p$, as in [12]; or in the nice dimensions with $p \geq n+1$, [4]; or certain quasihomogeneous germs with general n and p, see [6]. Another improvement is that the maps are allowed to be of general corank. The statements in [4] and [12] were for corank one maps. Damon's work was for general corank but the statements and proofs were mainly for quasihomogeneous germs with $n \geq p$. Furthermore, the classification given here is more constructive than Damon's. That is, given an $\mathcal{A}_e$-codimension one multi-germ one can show how it is constructed explicitly from mono-germs.

2 Augmentation

First we give some notation used throughout the paper. Standard notations can be found in [19]. If two germs f and g are $\mathcal{G}$-equivalent, for example $\mathcal{K}$- or $\mathcal{A}$-equivalent, then we use the notation $f \sim_{\mathcal{G}} g$. The set S will be a finite set of points and usually these will be the origins of a collection of copies of $\mathbb{C}^n$ for some n. The $\mathcal{A}_e$-codimension of a germ f will be denoted $\operatorname{cod}(f)$. Germs of complex analytic functions on $(\mathbb{C}^n, x)$ will be denoted by $\mathcal{O}_{n,x}$, or $\mathcal{O}_n$, if no confusion will result. The germs of *vector fields along a map* f will be denoted $\theta(f)$. The germs of *vector fields on* $(\mathbb{C}^n, x)$ will be denoted $\theta_{n,x}$ or θ_n, again if no confusion will result.

Remark 2.1 *It should be noted that if we add a submersive branch to a map f, then the $\mathcal{A}_e$-codimension of this new map is equal to* $\operatorname{cod}(f)$*. Hence, we shall assume throughout the paper* that our maps do not have submersive branches.

The results of this section were originally proved in [3], a more accessible reference for them is [4]. First we define the augmentation of a map-germ.

Definition 2.2 *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map with a 1-parameter stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}, S) \to (\mathbb{C}^p \times \mathbb{C}, 0)$, where $F(x, \lambda) = (f_\lambda(x), \lambda)$. Then the* augmentation of f by F *is the map $A_F(f)$ given by $(x, \lambda) \mapsto (f_{\lambda^2}(x), \lambda)$.*

Proposition 2.3 (See [4]) *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a finitely $\mathcal{A}$-determined map-germ with a 1-parameter stable unfolding. Then the following are true.*

(i). If f has $\mathcal{A}_e$-codimension one, then $A_F(f)$ has $\mathcal{A}_e$-codimension one.

(ii). The $\mathcal{A}$-equivalence class of $A_F(f)$ is independent of the choice of miniversal unfolding of f.

(iii). If $f \sim_{\mathcal{A}} f'$ and both have $\mathcal{A}_e$-codimension one, then $A(f) \sim_{\mathcal{A}} A(f')$.

Definition 2.4 *If f is not $\mathcal{A}$-equivalent to the augmentation of another germ, then f is called* primitive.

Since the augmentation of an $\mathcal{A}_e$-codimension one map is again codimension one we can augment repeatedly. Thus, define $A^m(f)$ to be the m-fold augmentation of f. For $m > 0$ this is the augmentation process repeated m times and for the trivial case $A^0(f) = f$.

An important lemma is the following, taken from [4] (Theorem 2.7) but originally proved in Proposition 2.5 of [3], see also [6] for a weaker version.

Lemma 2.5 (Diminishing Lemma) *Let g be an $\mathcal{A}_e$-codimension one multi-germ such that the miniversal unfolding of g is not $\mathcal{A}$-equivalent to the trivial unfolding of another map. Then, g is an augmentation of an $\mathcal{A}_e$-codimension one map-germ.*

Using this one can often prove results by reducing to the case of a primitive map.

3 Fundamental properties of $\mathcal{A}_e$-codimension one maps

One fundamental fact about $\mathcal{A}_e$-codimension one multi-germs (i.e. $|S| > 1$) is that the individual branches of the map are stable. In fact, we have the stronger statement from Corollary 5.7 of [4].

Theorem 3.1 *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a multi-germ of $\mathcal{A}_e$-codimension one, then for every proper subset S' of S, then the restriction of f to the multi-germ $f' : (\mathbb{C}^n, S') \to (\mathbb{C}^p, 0)$ is stable.*

Now we show that for primitive map-germs the $\mathcal{A}$-orbit is open in the $\mathcal{K}$-orbit. Then we apply this to prove that if two $\mathcal{A}_e$-codimension one maps (not necessarily primitive) are $\mathcal{K}$-equivalent, then they are $\mathcal{A}$-equivalent. (Also compare the following to Proposition 4.5.2(iii) of [19].)

Lemma 3.2 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is an $\mathcal{A}_e$-codimension one germ. Then, the $\mathcal{A}$-orbit of f is open in its $\mathcal{K}$-orbit if, and only if, f primitive.*

Proof. Denote the normal space of the $\mathcal{G}$-orbit of f by $N\mathcal{G}(f)$ and the extended one by $N\mathcal{G}_e(f)$.

It is easy to show, in the manner of the proof of Theorem 4.5.1 of [19], that if f has s branches then $\dim N\mathcal{K}(f) = \dim N\mathcal{K}_e(f) + s(n-p)$. In [20] it is shown that $\dim N\mathcal{A}(f) = \dim N\mathcal{A}_e(f) - s(p-n) + p$. A more accessible proof is given in [18].

The statement of the theorem is equivalent to the equality of the dimensions of the $\mathcal{K}$ and $\mathcal{A}$ normal spaces. We have

$$\begin{array}{rrcl} & \dim N\mathcal{A}(f) & = & \dim N\mathcal{K}(f) \\ \Longleftrightarrow & \dim N\mathcal{A}_e(f) - s(p-n) + p & = & \dim N\mathcal{K}_e(f) + s(n-p) \\ \Longleftrightarrow & 1+p & = & \dim N\mathcal{K}_e(f). \end{array}$$

Since f is primitive, its versal unfolding is a minimal stable map into a space of dimension $p+1$ and as $\mathcal{K}_e$-codimension is unchanged by unfolding, we have $\dim N\mathcal{K}_e(f) = p+1$, implying that the $\mathcal{A}$-orbit of f is open in its $\mathcal{K}$-orbit.

For the converse we note that, in general, the $\mathcal{K}$-orbit of a complex map has at most one open orbit. So, if an $\mathcal{A}_e$-codimension 1 map is not primitive, then it is $\mathcal{K}$-equivalent to a stable map, and the $\mathcal{A}$-orbit of this stable map is open in its $\mathcal{K}$-orbit. □

Now we come to the most fundamental property of codimension one germs. It resembles Mather's Theorem A of [15] on stable maps. A version for quasihomogeneous maps with $n \geq p$ can be deduced from [6].

Theorem 3.3 *Complex $\mathcal{A}_e$-codimension one maps that are $\mathcal{K}$-equivalent are $\mathcal{A}$-equivalent.*

Proof. Suppose that f and f' are $\mathcal{K}$-equivalent $\mathcal{A}_e$-codimension 1 germs. Since f and f' are codimension 1, the Diminishing Lemma gives that there exist primitive codimension one germs g and g' such that $f \sim A^m(g)$ and $f' \sim A^m(g')$, for some $m \geq 0$. The augmentations are both m-fold since the unfoldings F and F' of f and f' respectively will be stable and $\mathcal{K}$-equivalent since f and f' are (as $\mathcal{K}$-equivalence is unchanged by unfolding).

Since the unfoldings F and F' also unfold g and g' we deduce that g and g' are also $\mathcal{K}$-equivalent. There is at most one open $\mathcal{A}$-orbit in the $\mathcal{K}$-orbit of g and g', and so by Lemma 3.2 we can conclude that $g \sim_{\mathcal{A}} g'$.

From Proposition 2.3 we conclude that $A^m(g) \sim_{\mathcal{A}} A^m(g')$. □

Remarks 3.4 *(i). This closely resembles the stable case. The result is key to the classification as it is far easier to show $\mathcal{K}$-equivalence of maps than $\mathcal{A}$-equivalence.*

(ii). The theorem is not true over the reals. For example, the two real maps, $(x, y) \to (x, y^2, y^3 \pm x^2 y)$, have $\mathcal{A}_e$-codimension one, are $\mathcal{K}$-equivalent but are not $\mathcal{A}$-equivalent, see [16].

The theorem shows that if a stable map is the unfolding of an $\mathcal{A}_e$-codimension one map f, then f is the unique (up to $\mathcal{A}$-equivalence) $\mathcal{A}_e$-codimension one map associated to that stable. This greatly simplifies classification of germs. We take a map from a particular $\mathcal{K}$-class, produce the minimal stable unfolding from it using Mather's method and then look for an $\mathcal{A}_e$-codimension one germ associated to it in the pair of dimensions just below the stable map's pair of dimensions. The $\mathcal{A}_e$-codimension one germs with that $\mathcal{K}$-type will be augmentations of germs produced from this method.

Note that not all stable maps will have an $\mathcal{A}_e$-codimension one singularity associated to it. For example, Mather's $I_{a,b}$ singularity for $a, b \geq 3$, (this is given by $(x, y) \mapsto (xy, x^a + y^b)$), see [8]. Thus, once we have found an $\mathcal{A}_e$-codimension one associated to a stable map in a particular pair of dimensions it is unique, but for any given stable there may not be one to find.

Definition 3.5 *Suppose that a $\mathcal{K}$-class has an $\mathcal{A}_e$-codimension one map associated to it using the above method. Then, we say that* the $\mathcal{K}$-class supports an $\mathcal{A}_e$-codimension one singularity*.*

4 Reduction to linear slices

One would like a condition for recognizing when a stable map supports an $\mathcal{A}_e$-codimension one germ. We give a useful condition that reduces the problem to taking linear slices of the image.

The idea for the following theorem comes from [17], though one can see its antecedents in [6] and [7]. In [17] they assume that $n = 6$ and $p = 7$. This is unnecessary since all they use is that the stable map is minimal, i.e. not the trivial unfolding of something else. The theorem will be used to show that a stable map supports an $\mathcal{A}_e$-codimension 1 germ if, and only if, its liftable vector fields satisfy a condition on a linear slice of the image of the stable map.

First, we need some more notation and definitions. For maps $F : X \to Z$ and $g : Y \to Z$, let $g^*(F)$ denote the *pullback* of F by g. That is, the map defined on $\{(x, y) \in X \times Y \mid F(x) = g(y)\}$ that maps (x, y) to y.

Let $F : (\mathbb{C}^{n+1}, \widetilde{S}) \to (\mathbb{C}^{p+1}, 0)$ be a complex analytic map, then let $D(F)$ be the discriminant of F and $\mathrm{Derlog}(D(F))$ be the *liftable vector fields over* F. That is, all vector fields $\xi \in \theta_{p+1}$ such that there exists a vector field $\eta \in \theta_{n+1}$ with $dF(\eta) = \xi \circ F$.

Theorem 4.1 *Suppose $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a primitive $\mathcal{A}_e$-codimension one multi-germ. Then, the following hold.*

(i). The map f is $\mathcal{A}$-equivalent to a map induced as the pullback from a minimal stable $F : (\mathbb{C}^{n+1}, \widetilde{S}) \to (\mathbb{C}^{p+1}, 0)$ by a map $g : (\mathbb{C}^p, 0) \to (\mathbb{C}^{p+1}, 0)$.

(ii). The map g is an immersion.

(iii). The pullback $g^*(F)$ is $\mathcal{A}$-equivalent to $\widetilde{g}^*(F)$ where $\widetilde{g}$ is the linear part of g.

Proof. (i) Obvious: take a miniversal unfolding of f.

(ii) Let $T\mathcal{A}_e f$ be the $\mathcal{A}_e$-tangent space of f. If $g : (\mathbb{C}^p, 0) \to (\mathbb{C}^{p+1}, 0)$ induces f as a pullback, then by [5]

$$\frac{\theta(f)}{T\mathcal{A}_e f} \cong \frac{\theta(g)}{tg(\theta_p) + g^*(\mathrm{Derlog}(D(F)))},$$

where tg is the natural map from θ_p to $\theta(g)$ given by $tg(\chi) = dF \circ \chi$, and g^* is composition of the vector field with g, i.e $g^*(\eta) = \eta \circ g$.

Since F is minimal, i.e. not the trivial unfolding of any lower-dimensional germ, we have $\mathrm{Derlog}(D(F)) \subseteq \mathfrak{m}_{p+1}\theta_{p+1}$, where $\mathfrak{m}_{p+1}$ is the maximal ideal in $\mathcal{O}_{p+1}$. So

$$1 = \mathrm{cod}(f) \geq \dim_{\mathbb{C}} \frac{\theta(g)}{tg(\theta_p) + g^*(\mathrm{Derlog}(D(F))) + \mathfrak{m}_{p+1}\theta(g)} = \dim_{\mathbb{C}} \frac{\mathbb{C}^{p+1}}{\widetilde{g}(\mathbb{C}^p)} \geq 1.$$

Thus the inequalities are equalities and g is an immersion.

(iii) Let $g_t = (1-t)g + t\widetilde{g}$. By Nakayama's Lemma we have

$$tg_t(\mathfrak{m}_p\theta_p) + g_t^*(\mathrm{Derlog}(D(F))) = \mathfrak{m}_{p+1}\theta(g_t)$$

if, and only if,

$$tg_t(\mathfrak{m}_p\theta_p) + g_t^*(\mathrm{Derlog}(D(F))) + \mathfrak{m}_p^2\theta(g_t) = \mathfrak{m}_{p+1}\theta(g_t).$$

This just says that the codimension of $g_t^*(F)$ depends only on the linear part of g. Thus, the codimension of $g_t^*(F)$ is equal to 1 for all $t \in [0,1]$. By Mather's Lemma (Lemma 3.1 of [15]) the family members are all $\mathcal{A}$-equivalent. (Since all the maps are finitely $\mathcal{A}$-determined we can work in a suitable jet space in order to apply Mather's Lemma.) □

Now we can define a space that is isomorphic to the $\mathcal{A}_e$-tangent space of our codimension one map. The original definition in [7] is more general than this but the precise details are unimportant to us here.

Definition 4.2 *Suppose $F : (\mathbb{C}^n, \widetilde{S}) \to (\mathbb{C}^p, 0)$ is a stable map-germ and $h : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ is a complex analytic function. Then, the* extended ${}_V\mathcal{K}$-tangent space *of h is defined to be*

$$T_V\mathcal{K}_e(h) := \{\xi(h) \mid \xi \in \mathrm{Derlog}(D(F))\} + \langle h \rangle \mathcal{O}_p.$$

Suppose that f is a primitive $\mathcal{A}_e$-codimension one map-germ and then choose any minimal stable F with the same $\mathcal{K}$-class. From Theorem 4.1 we can induce f from F and g and we can assume g is linear. This image is defined by a linear map $h : (\mathbb{C}^{p+1}, 0) \to (\mathbb{C}, 0)$. Conversely, given a stable map and a linear map on its target we can construct an immersion with its image defined by the linear map. We can then pull back the stable map by this immersion. The consequence is that we are now reduced to finding primitive $\mathcal{A}_e$-codimension one maps by taking linear slices of discriminants (images) of minimal stable maps.

Theorem 4.3 (Condition for the existence of an $\mathcal{A}_e$-codimension one map) *Suppose that $F : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a minimal stable map-germ. Then, F supports an $\mathcal{A}_e$-codimension one map-germ if, and only if, there exists a linear germ $h : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ such that*

$$\langle \xi(h) \, : \, \xi \in \mathrm{Derlog}(D(F)) \rangle + \langle h \rangle = \mathfrak{m}_p \subseteq \mathcal{O}_p.$$

Furthermore, $h^{-1}(0)$ is the image of an immersion that pulls back via F to give the $\mathcal{A}_e$-codimension one germ.

Proof. As stated above, given a linear map $h : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ one can find an immersion $g : (\mathbb{C}^{p-1}, 0) \to (\mathbb{C}^p, 0)$ and vice versa. By the material in Section 5 of [7] we have

$$\begin{aligned} N\mathcal{A}_e(g^*(F)) &\cong N\mathcal{K}_{D(F),e}(g) \text{ by Theorem 2 of [5]} \\ &\cong \frac{\theta(h)}{T_V\mathcal{K}_e(h)} \text{ by Lemma 5.8 of [7]} \\ &\cong \frac{\mathcal{O}_p}{\langle \xi(h) \, : \, \xi \in \mathrm{Derlog}(D(F)) \rangle + \langle h \rangle}. \end{aligned}$$

Now, if h has the required property, then these isomorphisms give that $g^*(F)$ is $\mathcal{A}_e$-codimension 1, so F supports a codimension 1 map. Conversely, if F supports such a map, then by Theorem 4.1 it is the pullback of a linear immersion g, and in this case h has the required property. $\square$

5 Concatenation

There are two forms of concatenation, monic and binary, defined in [3] and [4]. These are operations to produce new multi-germs from other germs. Binary concatenation was defined for all pairs of dimensions; here one takes two germs to produce a new one. Monic concatenation was introduced for maps with $n \geq p-1$. Its effect was to take a germ and add an extra branch, with the branch giving rise to a non-singular component in the discriminant of the map. In [12] monic concatenation was defined for $n < p$, and again its effect is to take a germ and produce a new one with a non-singular component in the discriminant. In this section we combine the two approaches.

In [4] their definition of concatenation allowed the difference between the source and target dimensions of the maps to vary. To avoid overcomplicated statements and proofs we ask for this difference to be the same for all maps. In practice, this is not an unreasonable or overly restrictive assumption. The interested reader can easily adapt the definitions and proofs given here to overcome this complication.

It should be noted that Damon uses the essentially equivalent notion of product union in [6], but that his notion does not explicitly discriminate between monic and binary. This discrimination is not only helpful here but was used effectively in [13] when dealing with the topology of map-germs.

5.1 Monic concatenation

Monic concatenation produces a new germ by adding an immersive branch to the discriminant of the stable unfolding of a map.

Definition 5.1 *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a multi-germ of finite $\mathcal{A}_e$-codimension with a stable unfolding on the single parameter λ. Let $r = \max(0, p-n-1)$ and $k = -\min(0, p-n-1)$. Then, the* monic concatenation *of f with respect to F is the multi-germ $C_F(f) : (\mathbb{C}^{n+r+1}, S \amalg \{0\}) \to (\mathbb{C}^{p+r+1}, 0)$ given by*

$$\begin{aligned} (x, x', \lambda) &\mapsto (f_\lambda(x), x', \lambda) \\ (y, y') &\mapsto (y, 0, \dots, 0, \sum_{i=1}^{k} {y'_i}^2), \end{aligned}$$

where $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_r)$, $y = (y_1, \dots, y_{n+r+1})$ and $y' = (y'_1, \dots, y'_k)$.

Note that the zeroes in the lower branch above correspond to the coordinates x' in the upper branch.

The definition combines that of [4] (when the branches have equal source dimensions) and [12].

Thus, if $n < p$ we have

$$C_F(f) = \begin{cases} (x, x', \lambda) \mapsto (f_\lambda(x), x', \lambda) \\ \qquad\quad y \mapsto (y, 0, \dots, 0), \end{cases}$$

and if $n \geq p$ we have

$$C_F(f) = \begin{cases} \;\;(x, \lambda) \mapsto (f_\lambda(x), \lambda) \\ (y, y') \mapsto (y, \sum_{i=1}^{k} {y'_i}^2). \end{cases}$$

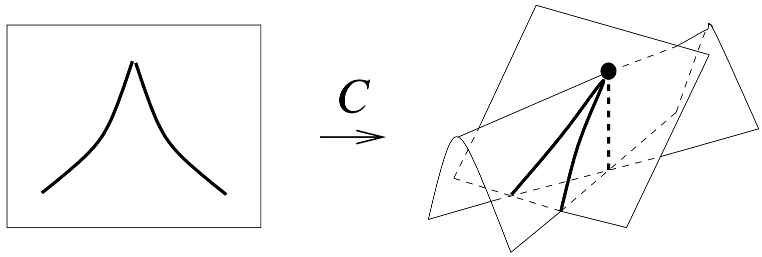

Figure 1: Monic concatenation of a cusp.

Example 5.2 *The real picture of the example of the monic concatenation of the standard cusp* $x \mapsto (x^2, x^3)$ *unfolded by* $(x^2, x^3 - \lambda x, \lambda)$ *is shown in Figure 1.*

The next proposition is analogous to statements in Proposition 2.3 on augmentation. See Theorem 3.3 of [4] and Proposition 4.2 of [12].

Proposition 5.3 *Suppose that* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$*, is a finitely* $\mathcal{A}$*-determined multi-germ with a 1-parameter stable unfolding. Then, the following are true.*

(i). The $\mathcal{A}_e$*-codimension of* $C_F(f)$ *is equal to the* $\mathcal{A}_e$*-codimension of* f*.*

(ii). If the $\mathcal{A}_e$*-codimension of* f *is one, then, under* $\mathcal{A}$*-equivalence,* $C_F(f)$ *is independent of the choice of the unfolding* F*. Hence, we use the notation* $C(f)$*.*

(iii). If f *has* $\mathcal{A}_e$*-codimension one and* $f \sim_{\mathcal{A}} f'$*, then* $C(f) \sim_{\mathcal{A}} C(f')$*.*

Proof. For $n \geq p$ the first statement is Theorem 3.1 of [4] (part (ii) is Theorem 3.3) and for $n < p$ it is Proposition 4.2 of [12].

The next two statements then follow from Theorem 3.3 as the maps involved are obviously $\mathcal{K}$-equivalent and are codimension 1 by part (i). □

For $\mathcal{A}_e$-codimension one germs we can apply this concatenation operation repeatedly without being concerned about the precise unfolding used and so we define $C^m(f)$ to be the m-fold concatenation of f, where $C^0(f) = f$.

5.2 Binary concatenation

Purely for reasons of exposition the following definition is slightly less general than in [4]. They allow the codimensions of the discriminants (in their ambient spaces) of two maps to be different. Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, and $g : (\mathbb{C}^{n'}, T) \to (\mathbb{C}^{p'}, 0)$, are finitely $\mathcal{A}$-determined map-germs with 1-parameter stable unfoldings and $p - n = p' - n'$. Let $F(x, \lambda) = (f_\lambda(x), \lambda)$ and $G(y, \mu) = (g_\mu(y), \mu)$ be the unfoldings of f and g respectively.

Definition 5.4 *The* binary concatenation *of* f *and* g *with respect to* F *and* G*, denoted* $B_{F,G}(f, g)$*, is the multi-germ* $B_{F,G}(f, g) : (\mathbb{C}^{n+p'+1}, S \amalg T) \to (\mathbb{C}^{p+p'+1}, 0)$ *given by*

$$\begin{array}{rcl} (X, x, u) & \mapsto & (X, f_u(x), u) \\ (y, Y, u) & \mapsto & (g_u(y), Y, u). \end{array}$$

It is difficult to draw real (non-schematic) pictures of binary concatenation. The only relevant example possible is the concatenation of two copies of the bi-germ mapping two distinct isolated points to the origin in $\mathbb{C}$. In Figure 2 it is shown that

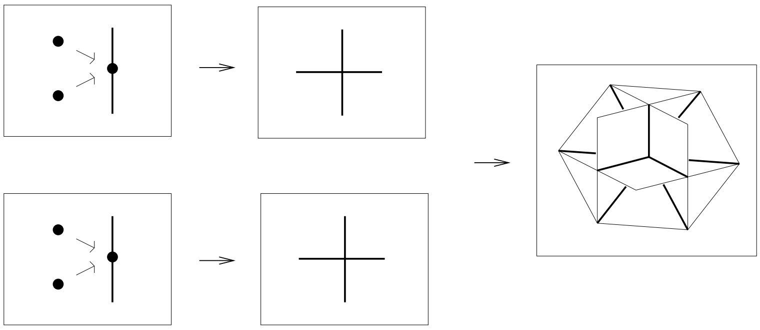

Figure 2: Binary concatenation of two simple bi-germs.

this bi-germ can be unfolded to give a map with the axes in $\mathbb{C}^2$ as its image. The two copies then combine to give a quadruple point.

In sharp contrast to the previous two process – augmentation and monic concatenation – we do not have the ideal situation in that if we take the binary concatenation of two $\mathcal{A}_e$-codimension one germs, then we do not necessarily produce an $\mathcal{A}_e$-codimension one germ. The following definition was introduced in [11] in a slightly different context, but is just what we need to define to circumvent the problem.

Definition 5.5 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a map-germ and F is a 1-parameter unfolding such that $F(x, \lambda) = (f_\lambda(x), \lambda)$ with $f_0 = f$. Then, F is called a* substantial unfolding *if λ is contained in the ideal $d\lambda(\mathrm{Derlog}(D(F))) \subseteq \mathcal{O}_p$.*

Examples 5.6 *(i). Suppose that f is quasihomogeneous and unfolds to F with an unfolding parameter λ which has non-zero weight. Then, the Euler vector field e is tangent to the discriminant. Since $d\lambda(e) = \mathrm{weight}(\lambda)\lambda \neq 0$, we deduce that F is a substantial unfolding.*

(ii). Suppose that $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable, corank one and $n < p$. Then F is a substantial unfolding of an $\mathcal{A}_e$-codimension one germ. See [12] where it is shown that corank one $\mathcal{A}_e$-codimension one germs are quasihomogeneous with a non-zero weight unfolding parameter.

Compare the following with 3.8 of [4] and Proposition 4.4 of [12].

Proposition 5.7 *Suppose that f and g are $\mathcal{A}_e$-codimension one. Then, the following are true.*

(i). ([4] Theorem 3.8) The map-germ $B_{F,G}(f, g)$ has $\mathcal{A}_e$-codimension one if, and only if, f or g has a substantial unfolding.

(ii). (C.f. [4] Proposition 3.11) If one of f or g has a substantial unfolding, then the map $B_{F,G}(f, g)$, up to $\mathcal{A}$-equivalence, is independent of the choice of unfoldings F and G. Hence, we use the notation $B(f, g)$.

(iii). Suppose at least one of f or g has a substantial unfolding. If $f \sim_{\mathcal{A}} f'$ and $g \sim_{\mathcal{A}} g'$, then $B(f', g') \sim_{\mathcal{A}} B(f, g)$.

Proof. (i) Theorem 3.8 of [4] states that $\mathrm{cod}(B_{F,G}(f, g)) \geq \mathrm{cod}(f) \times \mathrm{cod}(g)$ with equality if, and only if, one of them has a substantial unfolding (though the latter part is not phrased in that language). The statement obviously follows from this.

(ii) Suppose that F' and G' are other stable unfoldings. We can assume without loss of generality that F is a substantial unfolding. Since F and F' are stable and $\mathcal{K}$-equivalent they will be $\mathcal{A}$-equivalent and thus F' will be substantial.

Hence, $B_{F,G}(f,g)$ and $B_{F',G'}(f,g)$ have codimension 1. Since they are obviously $\mathcal{K}$-equivalent, by Theorem 3.3 they are $\mathcal{A}$-equivalent.

(iii) Again, the maps under consideration are $\mathcal{K}$-equivalent and codimension 1, so are $\mathcal{A}$-equivalent. □

One of the main difficulties in proving the main theorems of [4] arose on page 148. Suppose that $h = \{f, g\}$ is a primitive $\mathcal{A}_e$-codimension one map-germ, where f and g are collections of branches. Then, if f is transverse to the set along which g is $\mathcal{A}$-trivial, and vice versa, then one could *almost* get h to be a binary concatenation. In order to proceed they had to assume that f and g had a certain quasihomogeneity property. This was also done (more or less) in the corank 1 classification given in [12].

The next theorem will show that we can always get such an h into the binary concatenation form without making the extra quasihomogeneity assumption.

For any germ f we can define the *analytic stratum* as the set in the target along which f is trivial. (Trivial here means f is effectively a trivial unfolding of some map.) It is shown in Theorem 5.5 of [4] that this set is a manifold. Let $\tau(f)$ denote the tangent space to this manifold.

Theorem 5.8 *Suppose $h : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a primitive $\mathcal{A}_e$-codimension 1 map-germ and that $h = \{f, g\}$ for some germs f and g where $\tau(f)$ is transverse to g and $\tau(f)$ is transverse to g.*

Then, $h \sim_{\mathcal{A}} B(f_0, g_0)$ for some primitive $\mathcal{A}_e$-codimension one germs f_0 and g_0 and at least one of them has a substantial unfolding.

Proof. The hypotheses of the theorem imply that we are in the situation of Case 2 on page 148 of [4]. Thus, h is of the form

$$\begin{array}{rcl} (X, y, u) & \mapsto & (X, f_u(y), u) \\ (x, Y, u) & \mapsto & (g_{u,Y}(x), Y, u), \end{array}$$

where f_0 and $g_{0,0}$ are codimension 1 germs. By Lemma 5.18 of [4] these are primitive since h is. As stated earlier the point is that we almost have h in the form $B(f_0, g_{0,0})$. We now get round the problem by showing that one of the f_0 or $g_{0,0}$ has a substantial unfolding. For then, by Proposition 5.3(i) $B(f_0, g_{0,0})$ is codimension 1 and so, by Theorem 3.3, as it is $\mathcal{K}$-equivalent to h it must be $\mathcal{A}$-equivalent to h.

We can unfold h to get a stable map $\widetilde{H} = \{\widetilde{F}, \widetilde{G}\}$. By Mather's form of stable maps this is $\mathcal{A}$-equivalent to $H = \{F, G\}$, where F and G are trivial extensions of transverse stable maps:

$$\begin{array}{rcl} (u, v) & \mapsto & (u, \overline{F}(v)) \\ (w, z) & \mapsto & (\overline{G}(w), z). \end{array}$$

Thus, we can assume that $\tau(F) \oplus \tau(G) \cong \mathbb{C}^{p+1}$. Let $r = \dim_{\mathbb{C}} \tau(F)$ and $s = \dim_{\mathbb{C}} \tau(G)$

We have the commutative diagram:

$$\begin{array}{rcccc} H = \{F, G\} & : & \mathbb{C}^{n+1} & \to & \mathbb{C}^r \times \mathbb{C}^s \\ & & \uparrow \phi & & \uparrow \psi \\ \widetilde{H} = \{\widetilde{F}, \widetilde{G}\} & : & \mathbb{C}^{n+1} & \to & \mathbb{C}^{p+1} \\ & & \uparrow & & \uparrow \alpha \\ h = \{f, g\} & : & \mathbb{C}^n & \to & \mathbb{C}^p, \end{array}$$

where ϕ and ψ are diffeomorphisms and α is an immersion that induces f as a pullback from $\widetilde{H} = \{\widetilde{F}, \widetilde{G}\}$.

From Theorem 4.3 there exists a linear map $l : (\mathbb{C}^{r+s}, 0) \to (\mathbb{C}, 0)$ such that the pullback of $\{F, G\}$ by an immersion parametrizing the zero-set of l is $\mathcal{A}$-equivalent to $(\psi \circ \alpha)^*(\{F, G\})$. Furthermore, if we restrict to $\mathbb{C}^r$ then l induces a map $\mathcal{A}$-equivalent to f_0 (again via the pullback by an immersion), similarly l induces a map $\mathcal{A}$-equivalent to $g_{0,0}$ when we restrict to $\mathbb{C}^s$. Thus, $l = l_F + l_G$, where l_F and l_G induce codimension one maps.

Now, $\mathrm{Derlog}(D(H)) = \{\xi_1, \ldots, \xi_a, \eta_1, \ldots, \eta_b\}$ where $\{\xi_i\}$ are the vector fields liftable over F but not including the ones along the trivial part of F. Similarly, for $\{\eta_j\}$ and G.

The ${}_{D(H)}\mathcal{K}_e$-tangent space of l is

$$\langle \xi_i(l), \eta_j(l), l \rangle = \langle \xi_i(l_F), \eta_j(l_G), l_F + l_G \rangle.$$

This is obviously a subset of the ideal $\langle \xi_i(l_F), \eta_j(l_G), l_F, l_G \rangle$.

Since l, l_F and l_G all induce codimension 1 maps we get

$$\begin{aligned}
1 &= \dim \frac{\mathcal{O}_{r+s}}{\langle \xi_i(l), \eta_j(l), l \rangle} \\
&= \dim \frac{\mathcal{O}_{r+s}}{\langle \xi_i(l_F), \eta_j(l_G), l_F + l_G \rangle} \\
&\geq \dim \frac{\mathcal{O}_{r+s}}{\langle \xi_i(l_F), \eta_j(l_G), l_F, l_G \rangle} \\
&= \dim \frac{\mathcal{O}_r}{\langle \xi_i(l_F), l_F \rangle} \otimes_{\mathbb{C}} \frac{\mathcal{O}_s}{\langle \eta_j(l_G), l_G \rangle} \\
&= \dim \frac{\mathcal{O}_r}{\langle \xi_i(l_F), l_F \rangle} \times \dim \frac{\mathcal{O}_s}{\langle \eta_j(l_G), l_G \rangle} \\
&= 1.
\end{aligned}$$

Thus, the inequality above is an equality and so

$$\langle \xi_i(l_F), \eta_j(l_G), l_F + l_G \rangle = \langle \xi_i(l_F), \eta_j(l_G), l_F, l_G \rangle = \mathfrak{m}_{r+s}.$$

Hence, either $l_F \in \langle \xi_i(l_F) \rangle$ or $l_G \in \langle \eta_j(l_G), l_G \rangle$. That is, one of the codimension 1 germs has a substantial unfolding. So we can conclude that $B(f_0, g_{0,0})$ has codimension 1. $\square$

6 The three operations commute and the binary one is associative

In [12] it is shown for $\mathcal{A}_e$-codimension one maps with source dimension less than target dimension that various natural relations between the three operations A, B and C hold: They are pairwise commutative and binary concatenation is associative. In this section we show that the relations hold for all pairs of dimensions for $\mathcal{A}_e$-codimension one maps. This was done in [12] by specific coordinate changes in source and target and so the proofs were relatively long. The proofs given here are considerably shortened as the resulting maps are $\mathcal{K}$-equivalent and so are $\mathcal{A}$-equivalent.

We also show that for two particularly special bi-germs their binary concatenation with a germ f is $\mathcal{A}$-equivalent to the 2-fold monic concatenation of f.

Theorem 6.1 *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $g : (\mathbb{C}^{n'}, T) \to (\mathbb{C}^{p'}, 0)$ and $h : (\mathbb{C}^{n''}, T) \to (\mathbb{C}^{p''}, 0)$ are $\mathcal{A}_e$-codimension 1 multi-germs with $p'' - n'' = p' - n' = p - n$.*

(i). The A and C operations commute: $A(C(f)) \sim_{\mathcal{A}} C(A(f))$.

(ii). Suppose that $B(f,g)$ *or* $B(A(f),g)$ *have* $\mathcal{A}_e$*-codimension one. Then, the A and B operations commute:* $A(B(f,g)) \sim_{\mathcal{A}} B(A(f),g)$.

(iii). Suppose that $B(f,g)$ *or* $B(C(f),g)$ *have* $\mathcal{A}_e$*-codimension one. Then, the B and C operations commute:* $C(B(f,g)) \sim_{\mathcal{A}} B(C(f),g)$.

(iv). Suppose that $B(f,g)$*,* $B(g,h)$ *and* $B(B(f,g),h)$ *have* $\mathcal{A}_e$*-codimension one. Then, B is associative:* $B(f,B(g,h)) \sim_{\mathcal{A}} B(B(f,g),h)$.

Proof. In each part the maps on either side of the equivalence are obviously $\mathcal{K}$-equivalent and hence, if they are codimension one, then by Theorem 3.3 they are $\mathcal{A}$-equivalent.

To prove (i) we can first assume that $\mathrm{cod}(B(f,g)) = 1$. This implies that $AB(f,g)$ has codimension 1 by Proposition 2.3(i). It also implies that either f or g has a substantial unfolding. Thus either $A(f)$ or g has codimension 1. (It is easy to prove that f has a substantial unfolding if, and only if, $A(f)$ has one.) Hence, by Proposition 5.7(ii) $\mathrm{cod}(B(A(f),g)) = 1$. Thus, both maps are codimension 1. If we instead assume that $\mathrm{cod}(B(A(f),g)) = 1$, then either $A(f)$ or g has a substantial unfolding. Thus, either f or g has a substantial unfolding, and so $B(f,g)$ has codimension 1.

Parts (ii) and (iii) are proved in a similar fashion. □

We finish this section with two theorems, each of which gives a relation between B and C when one of the germs is a special bi-germ that will be of interest later.

Theorem 6.2 *Let* $g : (\mathbb{C}^n, \{0,0\}) \to (\mathbb{C}^{2n+1}, 0)$*,* $n \geq 0$*, be the* $\mathcal{A}_e$*-codimension one bi-germ*

$$\begin{array}{rcl} (w_1,\dots,w_n) & \mapsto & (w_1,\dots,w_n,0,\dots,0,0) \\ (z_1,\dots,z_n) & \mapsto & (0,\dots,0,z_1,\dots,z_n,0), \end{array}$$

(this gives two n-planes intersecting in a single point), and let $f : (\mathbb{C}^{n'}, S) \to (\mathbb{C}^{n'+n+1}, 0)$*,* $n' \geq 0$*, be an* $\mathcal{A}_e$*-codimension one multi-germ.*

Then $B(f,g) \sim_{\mathcal{A}} C^2(f)$*, and it has* $\mathcal{A}_e$*-codimension one.*

Proof. It is easy to calculate that the map g has a substantial unfolding, so $B(f,g)$ is codimension 1. Also, $C^2(f)$ is codimension 1 by Theorem 5.3. As both sides of the equivalence in the statement are codimension 1 and $\mathcal{K}$-equivalent they are $\mathcal{A}$-equivalent by Theorem 3.3. □

Theorem 6.3 *Let* $g : (\mathbb{C}^n, \{0,0\}) \to (\mathbb{C}, 0)$*,* $n \geq 1$*, be the* $\mathcal{A}_e$*-codimension one bi-germ*

$$\begin{array}{rcl} (w_1,\dots,w_n) & \mapsto & \displaystyle\sum_{i=1}^n w_i^2 \\ (z_1,\dots,z_n) & \mapsto & \displaystyle\sum_{i=1}^n z_i^2, \end{array}$$

and let $f : (\mathbb{C}^{n'+n}, S) \to (\mathbb{C}^{n'+1}, 0)$*,* $n' \geq 0$*, be an* $\mathcal{A}_e$*-codimension one multi-germ.*

Then $B(f,g) \sim_{\mathcal{A}} C^2(f)$*, and has* $\mathcal{A}_e$*-codimension one.*

Proof. This is the same as the proof of the previous theorem. Note that g is quasihomogeneous with a versal unfolding parameter of non-zero weight and so has a substantial unfolding. □

7 Classification of multi-germs

We know come to the main classification theorem. This allows us to describe all complex $\mathcal{A}_e$-codimension one multi-germs in terms of $\mathcal{A}_e$-codimension one mono-germs. Recall that we do not allow submersive branches.

Theorem 7.1 *Suppose that $h : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is an $\mathcal{A}_e$-codimension one germ. Then, h is $\mathcal{A}$-equivalent to one of the following distinct mappings:*

(i). $A^m C^s g$, (all branches of the discriminant of h are non-singular),

(ii). $A^m C^s f_0$, (exactly one branch of the discriminant of h is singular),

(iii). $A^m C^s B(f_1, B(f_2, B(f_3, \ldots, B(f_{t-1}, f_t))))$, (multiple branches of the discriminant of h are singular),

where $f_i : (\mathbb{C}^{n'}, 0) \to (\mathbb{C}^{n'+p-n}, 0)$ is a primitive $\mathcal{A}_e$-codimension one mono-germ; $m,\ s,\ n',\ n'+p-n \in \mathbb{N} \cup \{0\}$; and $t \geq 2$. Also,

- *for $n < p$, g is the bi-germ given by two $(p-n-1)$-planes intersecting at a single point in $2(p-n)-1$ space (as in Theorem 6.2);*
- *for $n \geq p$, g is the bi-germ given by two Morse functions from $(\mathbb{C}^{n-p+1}, 0)$ to $(\mathbb{C}, 0)$, (as in Theorem 6.3).*

A map of the form above such that $f_1, \ldots, f_t$ are substantial has $\mathcal{A}_e$-codimension one.

Proof. First, since all the maps are $\mathcal{K}$-inequivalent they are all distinct.

Next, through use of the Diminishing Lemma (Lemma 2.5), we deduce that h is $\mathcal{A}$-equivalent to $A^m h_1$ for some primitive codimension one germ h_1 and $m \geq 0$. If h is a mono-germ, then it is of the form $A^m C^0 h_1$, i.e. of type (ii) above. Hence, we can assume that h is a multi-germ, that is, $|S| > 1$.

Suppose that $h = \{f, g\}$. Then, without loss of generality, we have two cases to consider.

Case 1: The map g is not transverse to $\tau(f)$: This case has two sub-cases.

(1a) The maps f and g are transverse: If $n \geq p$, then by Proposition 5.16 of [4] $h_1 \sim C(h_2)$ for some codimension 1 map h_2. If $n < p$, then the same is true by Proposition 6.2 of [12].

(1b) The maps f and g are not transverse: For $n \geq p$ Proposition 5.16 of [4] shows that g is mono-germ and the branches are Morse functions as in Theorem 6.3. For $n < p$ Proposition 6.2 of [12] shows that g is the special bi-germ of Theorem 6.2.

Case 2: The map f is transverse to $\tau(g)$ and g is transverse to $\tau(f)$: By Theorem 5.8, h_1 is in the form $B(f_0, g_0)$ for some primitive codimension 1 maps f_0 and g_0.

So, in all cases we can reduce h_1 to the concatenation of other codimension 1 germs. Hence, we can apply the above again, until we reduce h_1 to the concatenation of mono-germs and, in Case 1, special bi-germs. Theorems 6.1, 6.2 and 6.3 tells us that we can write h in the form of the statement.

The partial converse of the statement is true by Propositions 2.3, 5.3 and 5.7. □

Remark 7.2 *The theorem shows that if a multi-germ has at least one singular branch in the discriminant, then the stable map associated to the $\mathcal{K}$-class of a subset of the singular branches must support an $\mathcal{A}_e$-codimension one germ.*

Corollary 7.3 *Suppose that $h : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is an $\mathcal{A}_e$-codimension one germ and, furthermore, that the primitive mono-germs used in its construction are $\mathcal{A}$-equivalent to quasihomogeneous maps. Then, h is $\mathcal{A}$-equivalent to a quasihomogeneous map.*

Proof. The A, B and C operations construct quasihomogeneous maps from quasihomogeneous components and so by induction the result is true. $\square$

This corollary allows us to deduce Theorem 7.1 of [4] where the map is of corank 1, $n \geq p-1$ and (n, p) is in Mather's nice dimensions. Furthermore, from the theorem we can deduce the multi-germ classifications in [1] (where $h : (\mathbb{C}^3, S) \to (\mathbb{C}^3, 0)$ is corank 1), [6] (where $n \geq p$ and h is quasihomogeneous in various cases, e.g. the branches have simple $\mathcal{K}$-type), and [12] (where $n < p$ and h is corank 1).

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