Topological Triviality of Families of Singular Surfaces

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1 Introduction

We study the topological triviality of families of singular surfaces in $\mathbb{C}^3$ parametrized by $\mathcal{A}$-finitely determined map germs.

Finitely determined map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can be approximated by stable maps f_s, and information on the topology of such approximations can be obtained in terms of data calculable from the original map germ f. In [15], D. Mond defines the 0-stable invariants, $T(f)$, the number of triple points of f_s, and $C(f)$, the number of Whitney umbrellas of f_s, and shows how to compute them in terms of f. These two invariants together with $\mu(D^2(f))$, the Milnor number of the double point locus, form a complete set of invariants for $\mathcal{A}$-simple germs ([15]).

The following natural extension of the above result was formulated by Mond in [16]. Let $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a one-parameter family of finitely determined map germs. Does the constancy of the invariants $T(f_t)$, $C(f_t)$ and $\mu(D^2(f_t))$ imply the topological triviality of the family? In this work we give a positive answer to this question as a consequence of the following Lê-Ramanujam type theorem, ensuring that the constancy of the Milnor number of the double point locus characterizes the topological triviality of the family:

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Theorem 1.1. *Let $F : (\mathbb{C}^2\times\mathbb{C}, 0) \to (\mathbb{C}^3\times\mathbb{C}, 0)$ be a one-parameter unfolding of an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. The following statements are equivalent:*

1. *$\mu(D^2(f_t))$ is constant for $t \in T$, where T is a small neighborhood of 0 in $\mathbb{C}$;*

2. *F is topologically trivial.*

We prove that if the unfolding F is μ-constant, in the sense that $\mu(D^2(f_t))$ is constant for $t \in T$, where T is a small neighborhood of 0 in $\mathbb{C}$, then F is excellent, as defined by T. Gaffney in [8]. An excellent unfolding is a Thom stratified mapping and we obtain the trivialization by integrating controlled vector fields tangent to the strata of the stratification of F given by the stable types in source and target.

A natural question is whether μ-constant in a one-parameter family F also implies the Whitney equisingularity of the family. This is indeed the case, and it follows as a consequence of the following theorem which completely describes the equisingularity of F in terms of the equisingularity of $D^2(F)$, the double point locus in source, that happens to be a family of reduced plane curves. There is a well-understood theory of equisingularity for plane curves, starting with the results of O. Zariski ([24]), further developed among others, by Zariski himself, H. Hironaka, M. Lejeune, Lê Dũng Tràng and B. Teissier (see [9, 20] for surveys on the subject and [7] for some interesting new developments). For families of reduced plane curves, it is well known that topological triviality, Whitney equisingularity and bilipschitz equisingularity are equivalent notions. It is very surprising that this is also true for families of singular surfaces in $\mathbb{C}^3$ parametrized by $\mathcal{A}$-finitely determined map germs, as shown by the following theorem:

Theorem 1.2. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a one-parameter unfolding of an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Then, the following are equivalent:*

1. *F is topologically trivial.*

2. *F is Whitney equisingular.*

3. *F is bilipschitz trivial.*

Theorems 1.1 and 1.2 will be proved in Theorems 6.2 and 7.3, respectively.

For other results related to the subject discussed in this paper, see for instance [11, 19].

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2 Previous results

We first review some results on the geometry and classification of singularities of surfaces in 3-space.

Definition 2.1. *Two map germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are $\mathcal{A}-$equivalent, denoted by $g \sim_{\mathcal{A}} f$, if there exist map germs of diffeomorphisms $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $k : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$, such that $g = k \circ f \circ h^{-1}$.*

Definition 2.2. *$f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is finitely determined ($\mathcal{A}$-finitely determined) if there exists a positive integer k such that for any g with $j^k g(0) = j^k f(0)$ we have $g \sim_{\mathcal{A}} f$.*

Theorem 2.3 ((Geometric criterion, Mather-Gaffney, 1975))**.** *A map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is finitely determined if, and only if, for every representative f (of f) there exist U, a neighborhood of 0 in $\mathbb{C}^n$, and V, a neighborhood of 0 in $\mathbb{C}^p$, with $f(U) \subset V$, such that for all $y \in V \setminus 0$, the set $S = f^{-1}(y) \cap \Sigma(f)$ is finite and $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$ is stable.*

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ. From the classical result of Whitney [23], we know that the stable singularities in these dimensions are transverse double points, triple points and cross-caps. In this case, the above theorem implies that $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is $\mathcal{A}$-finitely determined if and only if for every representative f there exists a neighborhood U of 0, such that the only singularities of $f(U) \setminus \{0\}$ are transverse double points.

A finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ has a versal unfolding

$$\begin{array}{rcl} F : (\mathbb{C}^2 \times \mathbb{C}^r, 0) & \longrightarrow & (\mathbb{C}^3 \times \mathbb{C}^r, 0) \\ (x, u) & \mapsto & (f_u(x), u). \end{array}$$

Given a representative of the map germ F, which we also denote by F, defined in a neighborhood $U \times W$ of $(0,0)$ in $\mathbb{C}^2 \times \mathbb{C}^r$, we can define the bifurcation set $B \subset \mathbb{C}^r$ of F, by $B = \{u \in W : f_u \text{ is not stable}\}$.

The set B is a proper algebraic subset of $\mathbb{C}^r$, hence its complement is a connected set in $\mathbb{C}^r$. Hence, for any $u, u' \in W \setminus B$, f_u and $f_{u'}$ are stable and $f_u \sim_{\mathcal{A}} f_{u'}$.

Notice that f_u has a finite number of cross-caps and triple points and these are analytic invariants of the original map germ.

Let f_u be a local stable perturbation of f, (known as the *disentanglement* of f). In [15] D. Mond defines the following 0-stable invariants of f:

$$\begin{aligned} C(f) &= \# \text{ of cross-caps of } f_u, \\ T(f) &= \# \text{ of triple-points of } f_u. \end{aligned}$$

Formulas to compute $C(f)$ and $T(f)$ as the codimension of certain algebras associated to f are given in [15].

3 Double point locus

To study the topology of $f(\mathbb{C}^2)$, in a small neighborhood of 0, one needs a third invariant associated to the double point locus of f which we now describe. Let $f : U \to \mathbb{C}^p$ be a holomorphic map, where $U \subset \mathbb{C}^n$ is an open subset and $n \leq p$. We define the *double point set of* f, denoted by $\tilde{D}^2(f)$, as the closure in $U \times U$ of the set

$$\{(x,y) \in U \times U : f(x) = f(y), x \neq y\}.$$

To choose a convenient analytic structure for the double point set $\tilde{D}^2(f)$, we follow the construction of [18] which is also valid for holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^p$ with $n \leq p$. Let us denote the diagonals in $\mathbb{C}^n \times \mathbb{C}^n$ and $\mathbb{C}^p \times \mathbb{C}^p$ by Δ_n and Δ_p respectively and denote the sheaves of ideals defining them by $\mathcal{I}_n$ and $\mathcal{I}_p$ respectively. We write the points of $\mathbb{C}^n \times \mathbb{C}^n$ as (x, x'). Then, for each $i = 1, \ldots, p$, it is clear that

$$f_i(x) - f_i(x') \in \mathcal{I}_n$$

where $f = (f_1, \ldots, f_p)$. Hence there exist $\alpha_{ij}(x, x')$, $1 \leq i \leq p$, $1 \leq j \leq n$, such that

$$f_i(x) - f_i(x') = \sum_{j=1}^{n} \alpha_{ij}(x, x')(x_j - x'_j).$$

If $f(x) = f(x')$ and $x \neq x'$, then clearly every $n \times n$ minor of the matrix $\alpha = (\alpha_{ij})$ must vanish at (x, x'). We denote by $\mathcal{R}_n(\alpha)$ the ideal in $\mathcal{O}_{\mathbb{C}^{2n}}$ generated by the $n \times n$ minors of α. Then we define the *double point ideal* as

$$\mathcal{I}^2(f) = (f \times f)^* \mathcal{I}_p + \mathcal{R}_n(\alpha).$$

It is easy to verify that $V(\mathcal{I}^2(f)) = \tilde{D}^2(f)$ and we call this complex space the *double point locus of* f. At a non-diagonal point (x, x'), $\mathcal{I}^2(f)$ is generated by the functions $f_i(x) - f_i(x')$. Moreover, the restriction of $\mathcal{I}^2(f)$ to the diagonal Δ_n is the ideal generated by the $n \times n$ minors of the Jacobian matrix of f, so that $\Delta_n \cap \tilde{D}^2(f)$ is just the singular locus of f. The following property of the double point locus is a consequence of [3].

Lemma 3.1. *The codimension of $\tilde{D}^2(f)$ is less than or equal to p. Moreover, if the codimension is p, then $\tilde{D}^2(f)$ is Cohen-Macaulay.*

Proposition 3.2. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ. Then, $\tilde{D}^2(f)$ is a reduced curve.*

Proof. By Lemma 3.1 $\tilde{D}^2(f)$ is Cohen-Macaulay, hence it is of pure dimension 1 and satisfies Serre's conditions (S_1), that is all associated primes of $\mathcal{O}_{\tilde{D}^2(f)}$ are minimal (see for example [14, p. 183]). On the other hand, by the Mather-Gaffney criterion for finite determinacy, there is a representative $f : U \to \mathbb{C}^3$ of f defined on some open neighborhood U of 0 in $\mathbb{C}^2$ such that $f|_{U \setminus \{0\}}$ is stable and $f^{-1}(0) = \{0\}$. Thus, $\tilde{D}^2(f) \setminus \{0\}$ is a smooth curve in U and therefore $\mathcal{O}_{\tilde{D}^2(f)}$ satisfies Serre's conditions (R_0), that is the localization of $\mathcal{O}_{\tilde{D}^2(f)}$ at every minimal prime is regular. Hence, since $\mathcal{O}_{\tilde{D}^2(f)}$ satisfies both Serre's conditions (R_0) and (S_1), $\tilde{D}^2(f)$ is reduced (see for example [14, p. 183]). $\square$

Assume that G is a finite group which acts linearly on $\mathbb{C}^N$. This action induces an analytic structure on the quotient $\mathbb{C}^N/G$ so that the local ring at a point $z \in \mathbb{C}^N$ is given by

$$\mathcal{O}_{N,z}^G = \{h \in \mathcal{O}_{N,z} : gh = h, \forall g \in G\}.$$

Assume now that $I \subset \mathcal{O}_{N,z}$ is a G-invariant ideal. Then G acts also on the germ of analytic set $X = V(I) \subset (\mathbb{C}^N, z)$ and gives again an analytic structure on X/G with local ring

$$\mathcal{O}_X^G = \{h \in \mathcal{O}_X : gh = h, \forall g \in G\},$$

where $\mathcal{O}_X = \mathcal{O}_{N,z}/I$, in such a way that X/G embeds naturally in $(\mathbb{C}^N/G, z)$. If I is generated by G-invariant functions $a_1, \ldots, a_r \in \mathcal{O}_{N,z}$, then

$$\mathcal{O}_X^G \equiv \mathcal{O}_{N,z}^G / I^G,$$

where I^G is the ideal in $\mathcal{O}_{N,z}^G$ generated by the same functions $a_1, \ldots, a_r$. Since $\mathcal{O}_X^G$ is in fact a subring of $\mathcal{O}_X$, we have that if X is reduced, then X/G is also reduced.

In our case, if f is a holomorphic map or map germ from $\mathbb{C}^2$ to $\mathbb{C}^3$, then the double point ideal $\mathcal{I}^2(f)$ is S_2-invariant, where we consider the action of the group S_2 on $\mathbb{C}^2 \times \mathbb{C}^2$ given by $\tau(x, x') = (x', x)$. In this way, we can define the quotient complex space or complex space germ $\tilde{D}^2(f)/S_2$. It is a well known fact that $\mathbb{C}^2 \times \mathbb{C}^2/S_2$ is isomorphic to $\mathbb{C}^2$ times a quadratic cone in $\mathbb{C}^3$. In particular, $\tilde{D}^2(f)/S_2$ embeds in $\mathbb{C}^5$.

If $f : U \subset \mathbb{C}^2 \to \mathbb{C}^3$ is stable, then $\tilde{D}^2(f)$ is a smooth curve and the quotient map $p : \tilde{D}^2(f) \to \tilde{D}^2(f)/S_2$ is a 2-fold branched covering. For a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, $\tilde{D}^2(f)/S_2$ is the germ of a reduced curve and $\pi : \tilde{D}^2(f) \to \tilde{D}^2(f)/S_2$ is a finite map germ, which is generically 2-to-1.

To complete the setup, let $D^2(f) = p_1(\tilde{D}^2(f)) \subset \mathbb{C}^2$ and $f(D^2(f)) \subset \mathbb{C}^3$, where $p_1 : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ is the projection on the first factor. For a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, p_1 is finite when restricted to $\tilde{D}^2(f)$, in fact it is 1-to-1, and thus $D^2(f)$ is the germ of a one-dimensional analytic set. Analogously, f is also finite when restricted to $D^2(f)$ (although in this case the map is 2-to-1 except at 0), and thus $f(D^2(f))$ is also a germ of a one-dimensional analytic set. In both cases we consider the reduced analytic structure, so that they become germs of reduced curves.

We have the following commutative diagram

$$\begin{array}{ccc} \tilde{D}^2(f) & \longrightarrow & \tilde{D}^2(f)/S_2 \\ \downarrow & & \downarrow \\ D^2(f) & \longrightarrow & f(D^2(f)), \end{array} \tag{1}$$

where the columns are 1-to-1.

Therefore, there are 3 double-point sets associated to the source: $\tilde{D}^2(f)$, $D^2(f)$ and $\tilde{D}^2(f)/S_2$, and each of them is a reduced curve with isolated singularity. The Milnor number of each of these sets, as defined in [4], is an analytic invariant of the singularity.

Theorem 3.3 ((Mond and Marar [18])). *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ, f_s a stable perturbation of f and $X_s = f_s(\mathbb{C}^2)$, then the following results hold.*

1. $\chi(X_s) = C(f) + T(f) + \mu(\tilde{D}^2(f)/S_2)$,

2. *X_s is simply connected and has the homotopy type of a wedge of 2-spheres. The number of spheres in the wedge (known as the* image Milnor number*) is*

$$\mu_\Delta(f) = C(f) - 1 + T(f) + \mu(\tilde{D}^2(f)/S_2).$$

4 Families of finitely determined map germs

Definition 4.1. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ. A* one-parameter unfolding of f *is a map germ $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ of the form $F(x, y, t) = (f_t(x, y), t)$ such that $f_t(0) = 0$, $f_0 = f$. We say that an unfolding F is a* stabilization of f *if there is a representative $F : U \times T \to \mathbb{C}^3 \times T$, where T and U are open neighborhoods of 0 in $\mathbb{C}$ and $\mathbb{C}^2$ respectively, such that $f_t : U \to \mathbb{C}^3$ is stable for all $t \in T \setminus \{0\}$.*

Since we are in the range of the nice dimensions in the sense of Mather, it is well known that a stabilization of a finitely determined map germ always exists.

Given an unfolding F of f, we can also define the double point locus $\tilde{D}^2(F)$ which is considered in $(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}, 0)$ instead of $(\mathbb{C}^3 \times \mathbb{C}^3, 0)$ and the other set germs $\tilde{D}^2(F)/S_2$ in $(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}/S_2, 0)$, $D^2(F)$ in $(\mathbb{C}^2 \times \mathbb{C}, 0)$ and $F(D^2(F))$ in $(\mathbb{C}^3 \times \mathbb{C}, 0)$.

Lemma 4.2. *([1, Lemma 1.1]) Let X be a one parameter deformation of a reduced space curve X_0. Then, the following conditions are equivalent:*

1. *X is a flat deformation of X_0;*

2. *X is Cohen-Macaulay;*

3. *X is of pure dimension 2.*

Proposition 4.3. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Then, the projections $\tilde{D}^2(F) \to (\mathbb{C}, 0)$, $D^2(F) \to (\mathbb{C}, 0)$ and $\tilde{D}^2(F)/S_2 \to (\mathbb{C}, 0)$ are flat deformations of $\tilde{D}^2(f)$, $D^2(f)$ and $\tilde{D}^2(f)/S_2$ respectively. Moreover, if F is a stabilization, then the projection $\pi : \tilde{D}^2(F) \to (\mathbb{C}, 0)$ is a smoothing.*

Proof. Let $\pi : \tilde{D}^2(F) \to (\mathbb{C}, 0)$ be the projection. Note that the second part of the proposition follows from the first one, since for each $t \neq 0$, $\tilde{D}^2(f_t) = \pi^{-1}(t)$ which is smooth if f_t is stable.

Let us show that $\pi : \tilde{D}^2(F) \to (\mathbb{C}, 0)$ is a flat deformation of $\tilde{D}^2(f)$. Note that $\tilde{D}^2(F)$ is Cohen-Macaulay of pure dimension 2 by Lemma 3.1. We also know by Proposition 3.2 that $\tilde{D}^2(f)$ is reduced and has pure dimension 1. Thus, $\tilde{D}^2(F)$ is a Cohen-Macaulay 1-parameter deformation of the reduced space curve $\tilde{D}^2(f)$. Hence the result follows by Lemma 4.2.

The proof for $\tilde{D}^2(F)/S_2$ is analogous. Suppose that $\mathcal{I}^2(F)^{S_2}$ is generated by S_2-invariant functions $H_1, \dots, H_r \in \mathcal{O}_5$. Then $h_1, \dots, h_r \in \mathcal{O}_4$ are also S_2-invariant, where $h_i(x, x') = H_i(x, x', 0)$, and generate $\mathcal{I}^2(f)^{S_2}$. Hence, $\mathcal{I}^2(F)^{S_2}$ restricted to $t = 0$ is equal to $\mathcal{I}^2(f)^{S_2}$, that is $\tilde{D}^2(F)/S_2$ is a 1-parameter deformation of the space curve $\tilde{D}^2(f)/S_2$. On the other hand, since $\tilde{D}^2(F)$ is Cohen-Macaulay by Lemma 3.1 we also have that $\tilde{D}^2(F)/S_2$ is Cohen-Macaulay (see for example [2, Corollary 6.4.6]). Thus, $\tilde{D}^2(F)/S_2$ is a Cohen-Macaulay 1-parameter deformation of the reduced space curve $\tilde{D}^2(f)/S_2$. Hence the result follows by Lemma 4.2.

Finally, the flatness of $D^2(F)$ follows from the fact that $D^2(F)$ is a hypersurface in $\mathbb{C}^3$, hence it is Cohen-Macaulay. Therefore, $\tilde{D}^2(F)$ is a Cohen-Macaulay 1-parameter deformation of the reduced plane curve $\tilde{D}^2(f)$. Hence the result follows by Lemma 4.2. ∎

Definition 4.4. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. We will say that F is μ-constant if $\mu(D^2(f_t))$ is constant along T, for $t \in T$, where T is a small neighborhood of 0 in $\mathbb{C}$.*

Definition 4.5. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. We say that F is* topologically trivial *if there are homeomorphism map germs:*

$$\Phi : (\mathbb{C}^2 \times \mathbb{C}, (0,0)) \to (\mathbb{C}^2 \times \mathbb{C}, (0,0)), \quad \Phi(x,t) = (\phi_t(x), t),\ \phi_0(x) = x,\ \phi_t(0) = 0,$$
$$\Psi : (\mathbb{C}^3 \times \mathbb{C}, (0,0)) \to (\mathbb{C}^3 \times \mathbb{C}, (0,0)), \quad \Psi(y,t) = (\psi_t(y), t),\ \psi_0(y) = y,\ \psi_t(0) = 0,$$

such that $F = \Psi \circ G \circ \Phi^{-1}$*, where* $G(x,t) = (f(x),t)$ *is the trivial unfolding of* f*.*

The following definitions were given by T. Gaffney in [8] for finitely determined map germs in dimensions (n,p). We restrict ourselves to the case $(n,p) = (2,3)$.

Definition 4.6. *We say that* F *is a* good unfolding *of* f *if there exist neighborhoods* U *of* 0 *in* $\mathbb{C}^2 \times \mathbb{C}$ *and* W *of* 0 *in* $\mathbb{C}^3 \times \mathbb{C}$*, such that the following hold.*

(i) $F^{-1}(\{0\}\times T) = \{0\}\times T$*, that is,* F *maps* $U\setminus(\{0\}\times T)$ *into* $W\setminus(\{0\}\times T)$*.*

(ii) For all $(z_0,t_0) \in W\setminus(\{0\}\times T)$ *the map* $f_{t_0} : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ *is stable, where* $S = F^{-1}(z_0,t_0) \cap \Sigma F \cap U$ *(which is a finite set) and* ΣF *denotes the singular set of the unfolding* F*).*

Definition 4.7. *If there exists a curve* $\alpha(s) = (x(s), y(s), t(s))$ *in* $(\mathbb{C}^2\times\mathbb{C}, 0)$*, containing* 0 *in its closure, such that* $(x(s), y(s))$ *is a cross-cap point of* $f_{t(s)}$*, then we say that* F has coalescing of cross-cap singularities.

We can make an analogous definition for coalescing of triple points. The following proposition is a particular case of Proposition 3.6 in Gaffney's paper.

Proposition 4.8. *Let* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *be a finitely determined map germ and* $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ *be a good unfolding of* f*. Then, the following conditions are equivalent:*

(i) $C(f_t)$ *and* $T(f_t)$ *are constant.*

(ii) F *has no coalescing of cross-caps or triple points.*

Definition 4.9. *A one-parameter unfolding* F *of* f *is an* excellent unfolding *if it is good and the* 0*-stable invariants* $C(f_t)$ *and* $T(f_t)$ *remain constant.*

Excellent unfoldings have a natural stratification. In the source there are the following strata:

$$\left\{\mathbb{C}^2 \setminus D^2(F), D^2(F) \setminus (\{0\} \times T), \{0\} \times T\right\}.$$

In the target, the strata are:

$$\left\{\mathbb{C}^3 \setminus F(\mathbb{C}^2 \times \mathbb{C}), F(\mathbb{C}^2 \times \mathbb{C}) \setminus \overline{F(D^2(F))}, F(D^2(F)) \setminus (\{0\} \times T), \{0\} \times T\right\}.$$

Notice that F preserves the stratification, that is, F sends a stratum into a stratum.

To finish the section, we recall the main result in T. Gaffney's paper [8], which gives, for any pair of dimensions (n, p), the topological triviality of excellent unfoldings of finitely determined map germs for which the polar invariants associated to the stable types in source and target remain constant.

Theorem 4.10 ((Theorem 7.1, [8])). *Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ be an excellent unfolding. Let us assume that all polar invariants of all stable types which occur in the stratification associated to F are constant at the origin. Then, the unfolding is topologically trivial.*

For finitely determined map germs in the nice dimensions or their boundary, Gaffney also proved ([8], Theorem 7.3) that an excellent unfolding F is Whitney equisingular if, and only if, all polar invariants of all stable types of F are constant.

5 Properties of μ-constant unfoldings

Theorem 5.1. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a topologically trivial family of finitely determined map germs. Then,*

(a) the Milnor numbers $\mu(D^2(f_t))$, $\mu(\tilde{D}^2(f_t))$, and $\mu(\tilde{D}^2(f_t)/S_2)$ are constant;

(b) the 0-stable invariants $C(f_t)$ and $T(f_t)$ are constant.

In particular, F is a μ-constant unfolding of f.

Proof. The following formulas were proved by Marar and Mond in [18], Theorem 3.4:

$$\mu(D^2(f_t)) = \mu(\tilde{D}^2(f_t)) + 6T(f_t) = 2\mu(\tilde{D}^2(f_t)/S_2) + C(f_t) + 6T(f_t) - 1.$$

From these it suffices to prove that $\mu(\tilde{D}^2(f_t))$, $\mu(\tilde{D}^2(f_t)/S_2)$ and $\mu(D^2(f_t))$ are constant. (We also use the fact that all of the above invariants are upper semi-continuous.)

Since $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ defined by $F(x, y, t) = (f_t(x, y), t)$ is topologically trivial, there are homeomorphisms $\Phi : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^2 \times \mathbb{C}, 0)$ and $\Psi : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$, as in the Definition 4.3, such that $\Psi \circ F \circ \Phi^{-1} = G$, where $G(x, y, t) = (f(x, y), t)$ is the trivial unfolding of f.

It is easy to prove that these homeomorphisms give rise to homeomorphisms (respecting the S_2-action) on the double point locus:

1. $\tilde{\Phi}^2 : \tilde{D}^2(F) \to \tilde{D}^2(G)$,
2. $\overline{\Phi}^2 : \tilde{D}^2(F)/S_2 \to \tilde{D}^2(G)/S_2$,
3. $\Phi^2 : D^2(F) \to D^2(G)$,

which obviously commute with the projections to $\mathbb{C}$.

Clearly, $\tilde{D}^2(G)$ is homeomorphic to $\tilde{D}^2(f) \times T$, $\tilde{D}^2(G)/S_2$ to $\tilde{D}^2(f)/S_2 \times T$ and $D^2(G)$ to $D^2(f) \times T$.

Hence, the above morphisms are topological trivializations of flat families of reduced curves with isolated singularities. Therefore, by Theorem 5.2.2 in [4] we have that $\mu(D^2(f_t))$, $\mu(\tilde{D}^2(f_t))$ and $\mu(\tilde{D}^2(f_t)/S_2)$ are constant. $\square$

Next we will prove that μ-constant unfoldings are excellent.

Theorem 5.2. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be an unfolding of a finitely determined map germ f. Then, the following statements are equivalent:*

1. *$\mu(D^2(f_t))$ is constant (i.e., F is a μ-constant unfolding);*
2. *$\mu(\tilde{D}^2(f_t))$ and $T(f_t)$ are constant;*
3. *$\mu(\tilde{D}^2(f_t)/S_2)$, $C(f_t)$ and $T(f_t)$ are constant;*
4. *The image Milnor number μ_Δ is constant.*

Moreover, any of these conditions implies that F is an excellent unfolding.

Proof. The equivalence of the four conditions above comes from Theorem 3.3 and from the formulas from Marar and Mond quoted in the proof of the previous theorem.

Let us assume now that $\mu(D^2(f_t))$ is constant in the deformation. To prove that F is excellent, we proceed as in [8], Theorem 8.7.

To verify the first condition of goodness, we assume by contradiction that, in any neighborhood of 0×0 in $\mathbb{C}^2 \times \mathbb{C}$, we have $F^{-1}(\{0\} \times T) \neq \{0\} \times T$. If the points in $F^{-1}(\{0\} \times T) \setminus (\{0\} \times T)$ lie in the singular set $\Sigma(F)$, of F, then $C(f_t)$ must change at the origin, so we can assume they lie in $F^{-1}(t) \setminus (\{0\} \times T \cup \Sigma(F))$.

Consider the intersection of $f_t(\mathbb{C}^2_x)$ with $f_t(\mathbb{C}^2_0)$ where $(x,t) \in F^{-1}(\{0\} \times T) \setminus (\{0\} \times T)$. If the intersection lies in $\Sigma(f_t(\mathbb{C}^2_0))$ then $T(f_t)$ is at least one dimensional. Hence f would not be finitely determined if this holds for all t close to 0. If $f_t(\mathbb{C}^2_x)$ meets $\Sigma(f_t(\mathbb{C}^2_0))$ properly, then $D^2(f_t)_x$ must have a singularity, hence $\mu(D^2(f_t))$ must jump at 0.

Considering the second condition for F to be good, we suppose it fails, so there exists an arc of points $(z(t), t)$ in $\mathbb{C}^3 \times \mathbb{C}$, with $(0,0)$ in its closure such that f_t is not a stable map germ on $f_t^{-1}(z(t))$. If $f_t^{-1}(z(t))$ consists of a single point or three or more points, then $z(t) \in \overline{f_t(\Sigma(f_t)) \cup f_t(T(f_t))}$, so either $C(f_t)$ or $T(f_t)$ jumps at the origin. The only possibility not so eliminated is that $f_t^{-1}(z(t))$ consists of two points, say x_1 and x_2 with $x_1 \neq x_2$, at which f_t is an immersion and $f(x_1) = f(x_2)$.

We now show that x_1 and x_2 are singular points of $D^2(f_t)$. The easiest way to see this is by picking disjoint neighborhoods U_i of x_i and choosing coordinates centered at x_i so that $f_t|_{U_1} = (x_1, y_1, 0)$ and $f_t|_{U_2} = (x_2, y_2, f_3(x_2, y_2))$ with $\operatorname{grad} f_3(0,0) = 0$. Then the equations for $\tilde{D}(f_t)$ are

$$\begin{cases} x_1 - x_2 &= 0, \\ y_1 - y_2 &= 0, \\ f_3 &= 0. \end{cases}$$

So $\tilde{D}^2(f_t)$ has a singularity at $(0,0) \times (0,0)$ and $\tilde{D}^2(f_t)$ sits in the diagonal of $U_1 \times U_2$. Since the projection onto either factor is an isomorphism when restricted to the diagonal, $D^2(f_t)$ in $\mathbb{C}^2$ has a singularity at x_i. Hence the constancy of $\mu(D^2(f_t))$ implies that such an arc can not exist in this case either. $\square$

6 The main result

Lemma 6.1. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a μ-constant unfolding of a finitely determined map germ f. Then, the families of curves $D^2(F)$, $\tilde{D}^2(F)$, and $\tilde{D}^2(F)/S_2$ are Whitney equisingular along $\{0\} \times T$.*

Proof. Since $\mu(D^2(f_t))$ is constant and $D^2(F)$ is a flat family of reduced plane curves we have that $(D^2(F) \setminus (\{0\} \times T), \{0\} \times T)$ is Whitney regular. In particular, the $D^2(f_t)$ have constant multiplicity at 0 (see [4]). On the other hand, by Theorem 5.2 we have that the flat family $\tilde{D}^2(F)$ of reduced curves have constant Milnor numbers and, since $p_1 : \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2 \times \mathbb{C}$ is a submersion which restricts to a 1-to-1 map $p_1 : \tilde{D}^2(F) \to D^2(F)$, we also have that the family $\tilde{D}^2(F)$ has constant multiplicity at 0. Hence, by a result of Buchweitz and Greuel [4] $(\tilde{D}^2(F) \setminus (\{0\} \times T), \{0\} \times T)$ is Whitney regular. Analogously, $\tilde{D}^2(F)/S_2$ is also a flat family of reduced space curves which has constant Milnor number and constant multiplicity, because $p : \tilde{D}^2(F) \to \tilde{D}^2(F)/S_2$ is 2-to-1 and the source has constant multiplicity. Hence, $(\tilde{D}^2(F)/S_2 \setminus (\{0\} \times T), \{0\} \times T)$ is Whitney regular.

□

Theorem 6.2. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a one-parameter unfolding of a finitely determined map germ f. The following statements are equivalent:*

1. *$\mu(D^2(f_t))$ is constant for $t \in T$, where T is a small neighborhood of 0 in $\mathbb{C}$;*

2. *F is topologically trivial.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Theorem 5.1.

The idea of the proof of the converse statement is to construct integrable vector fields ξ in $\mathbb{C}^2 \times \mathbb{C}$ and η in $\mathbb{C}^3 \times \mathbb{C}$, tangent to the strata of the stratifications in source and target respectively, such that

$$dF(\xi) = \eta \circ F.$$

In this way, the integral curves of ξ and η will define the families of homeomorphisms $h_t : \mathbb{C}^2 \to \mathbb{C}^2$, $k_t : \mathbb{C}^3 \to \mathbb{C}^3$ such that

$$k_t \circ f_t \circ h_t^{-1} = f.$$

Since $D^2(F)$ is Whitney equisingular along $\{0\} \times T$ as a family of curves in $\mathbb{C}^2 \times \mathbb{C}$, it follows from the First Isotopy Lemma [10] that the vector field $V_0 = \frac{\partial}{\partial t}$ on $\{0\} \times T$ lifts to an integrable stratified vector field V in a neighborhood U of 0 in $\mathbb{C}^2 \times \mathbb{C}$. The restriction of V to each stratum is smooth and tangent to the stratum, $d\pi_T(V) = V_0$ and $d\rho(V) = 0$, where

$\pi_T : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$ is the projection on the t-axis and ρ is a control function, $\rho : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$, $\rho \geq 0$, $\rho^{-1}(0) = \{0\} \times T$ (see [10] for more details).

Notice that $D^2(F)$ has an analytic S_2-action given by $\sigma(x, y, t) = (x', y', t)$, where (x', y', t) is the unique point of $D^2(F)$ such that $F(x, y, t) = F(x', y', t)$. We can average the vector field V to obtain a new controlled vector field on $D^2(F)$,

$$W(x, y, t) = \frac{V(x, y, t) + V(x', y', t)}{2},$$

satisfying $W(x, y, t) = W(x', y', t)$, whenever $F(x, y, t) = F(x', y', t)$ and $d\pi_T(W) = \frac{\partial}{\partial t}$. Notice that the function $\rho^*(x, y, t) = \frac{\rho(x,y,t)+\rho(x',y',t)}{2}$, is a control function for W, that is $d\rho^*(W) = 0$.

We can extend this vector field to a controlled vector field ξ defined in a neighborhood of 0 in $\mathbb{C}^2 \times \mathbb{C}$, (for example using the Kuo vector field, [12]).

The vector field η given by $\eta = dF(\xi)$ is a well defined integrable vector field in the image $F(D^2(F))$. Furthermore, since F is an isomorphism outside the double point locus, η can be extended to the whole image $F(\mathbb{C}^2 \times \mathbb{C})$. Certainly we can extend this vector field to the ambient space giving the desired topological trivialization. $\square$

7 Bilipschitz triviality of μ-constant unfoldings

A mapping $\phi : U \subset K^n \to K^p,\ K = \mathbb{R}$ or $\mathbb{C}$, is called *Lipschitz* if there exists a constant $c > 0$ such that:

$$\|\phi(x) - \phi(y)\| \leq c\|x - y\| \ \forall\ x, y \in U.$$

When $n = p$ and ϕ has a Lipschitz inverse, we say that ϕ is *bilipschitz*.

Two map germs $f, g : (K^n, 0) \to (K^p, 0)$ are called *bilipschitz equivalent* if there exists a bilipschitz map-germ $\phi : (K^n, 0) \to (K^n, 0)$ such that $f = g \circ \phi$.

A one-parameter unfolding $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is *bilipschitz trivial* if it is topologically trivial as in Definition 4.3 and the families of homeomorphisms in source and target are families of bilipschitz homeomorphisms.

When the bilipschitz trivialization is obtained by integrating bilipschitz vector fields, we say that F is *strongly bilipschitz trivial*.

Before proving the next theorem, we state a result of McShane ([13]), which is a weak version of a Theorem of Kirszbraun on the extension of Lipschitz functions:

Theorem 7.1. *([13]) Let $X \subset \mathbb{R}^n$ be a metric subspace of the Euclidean space and $f : X \to \mathbb{R}$ be a c-Lipschitz mapping, that is, $||f(x) - f(y)|| \leq c||x - y||$. Then,*

$$F(z) = sup_{\{x \in X\}}(f(x) - c||x - z||),\ z \in \mathbb{R}^n$$

is a c-Lipschitz extension of f.

Remark 7.2. If f depends continuously on parameters $t = (t_1, \ldots, t_s)$, i.e., $f(x, t)$, $x \in X$ is continuous in (x, t) and is c-Lipschitz in x, with the constant c not depending on t, then it follows from the above result that there exists a Lipschitz extension $F(x, t)$, $x \in \mathbb{R}^n$. Moreover, if v is a vector field on X, depending continuously on parameters, we can also apply Theorem 7.1 to each coordinate function of v to obtain an extension $c\sqrt{n}$-Lipschitz V to $\mathbb{R}^n$.

Theorem 7.3. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a one-parameter unfolding of an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Then, the following are equivalent:*

1. *F is topologically trivial.*
2. *F is Whitney equisingular.*
3. *F is bilipschitz trivial.*

Proof. To get the result it is sufficient to prove that (a) implies (c), since the conditions $(c) \implies (b) \implies (a)$ hold in general. We do this following the same steps as in Theorem 1.1, constructing Lipschitz vector fields ξ in $\mathbb{C}^2 \times \mathbb{C}$, and η in $\mathbb{C}^3 \times \mathbb{C}$ such that $dF(\xi) = \eta \circ F$.

As the set $D^2(F)$ is a family of reduced plane curves, then the following are equivalent (cf. [4, 24]):

(1) $\mu(D^2(f_t))$ is constant;

(2) $D^2(F)$ is topologically trivial;

(3) $D^2(F)$ is Whitney equisingular;

(4) $D^2(F)$ is strongly bilipschitz trivial.

As in Theorem 1.1, there is a vector field V defined in a neighborhood of the origin in $\mathbb{C}^2 \times \mathbb{C}$, $d\pi_T(V) = \frac{\partial}{\partial t}$. Since $D^2(F)$ satisfies (4), we now assume that V is a Lipschitz vector field.

We now lift V to a controlled vector field $\tilde{V}$ in $(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}, 0)$, which is tangent to the strata of $\tilde{D}^2(F)$. Clearly, $d\pi_T(\tilde{V}) = \frac{\partial}{\partial t}$.

Since the set $\tilde{D}^2(F)$ is S_2-invariant, we can average the vector field $\tilde{V}$ to obtain a new controlled vector field

$$\tilde{W}(x, y, x', y', t) = \frac{\tilde{V}(x, y, x', y', t) + \tilde{V}(x', y', x, y, t)}{2},$$

tangent to the strata of the stratification in $(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}, 0)$, which is invariant under the S_2 action on $\tilde{D}^2(F)$.

Let $p_i : (\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^2 \times \mathbb{C}, 0)$, $i = 1, 2$, be the canonical projections: $p_1(x, y, x', y', t) = (x, y, t)$ and $p_2(x, y, x', y', t) = (x', y', t)$. The restriction of p_1 to $\tilde{D}^2(F)$ is a 1-to-1 generic projection in the sense that the limit of all secant lines in $\tilde{D}^2(F)$ does not live in the kernel of p_1. Then, it follows from [21], page 354, (see also [6]) that $p_1^{-1} : D^2(F) \to \tilde{D}^2(F)$ is a bilipschitz homeomorphism. Then, $p_2 \circ p_1^{-1} : D^2(F) \to D^2(F)$ is also a bilipschitz homeomorphism. Notice that $p_2 \circ p_1{}^{-1}$ is the map-germ σ defined in the proof of Theorem 1.1 and which gives the S_2 action in $D^2(F)$.

Then the vector field W defined in $D^2(F)$ by $W(x, y, t) = \tilde{W}(x, y, \sigma(x, y, t), t)$, is a Lipschitz vector field in $D^2(F)$. We now use McShane's result to obtain a Lipschitz vector field ξ in $\mathbb{C}^2 \times \mathbb{C}$ which extends W. The rest of the proof follows as in Theorem 6.3, noticing that η is clearly a Lipschitz vector field, and that all the necessary extensions can be made Lipschitz, applying again McShane's result.

□

Remark 7.4. It is clear that Theorem 1.1 could be obtained as a corollary of Theorem 7.3. However, we present Theorem 1.1 as our main result, because it has a more general setting, while Theorem 7.3 express properties which do not hold in other dimensions.

As a consequence of the above theorem, it follows that the rich theory of invariants of plane curves translate into results for singular surfaces parametrized by finitely determined map germs. We restrict ourselves to the following corollary, which follows from the above result, and Gaffney's Theorem 7.1, in [8].

Corollary 7.5. *Let F be as before and $m_0(f_t)$ be the multiplicity of f_t at zero. If F is topologically trivial, then $m_0(f_t)$ is constant.*

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