Mathematical writing

An undergraduate course

Franco Vivaldi
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Mathematical Writing students commented on the "unexpected depth" required of their thinking, when asked to offer verbal explanations.

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- This task requires understanding of structure, and organisation;
- it will give students an opportunity to express their knowledge, intelligence, and individuality;
- but it will also expose immaturity, logical faults, and incompetence in basic mathematics.

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4. If your point lies outside, on, or inside the circle, respectively, you will obtain two distinct real solutions, two identical solutions, or two complex solutions for the slope, respectively. Any geometrical configuration involving vertical lines (infinite slope) will require some care.

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Teacher **Student**

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theorems

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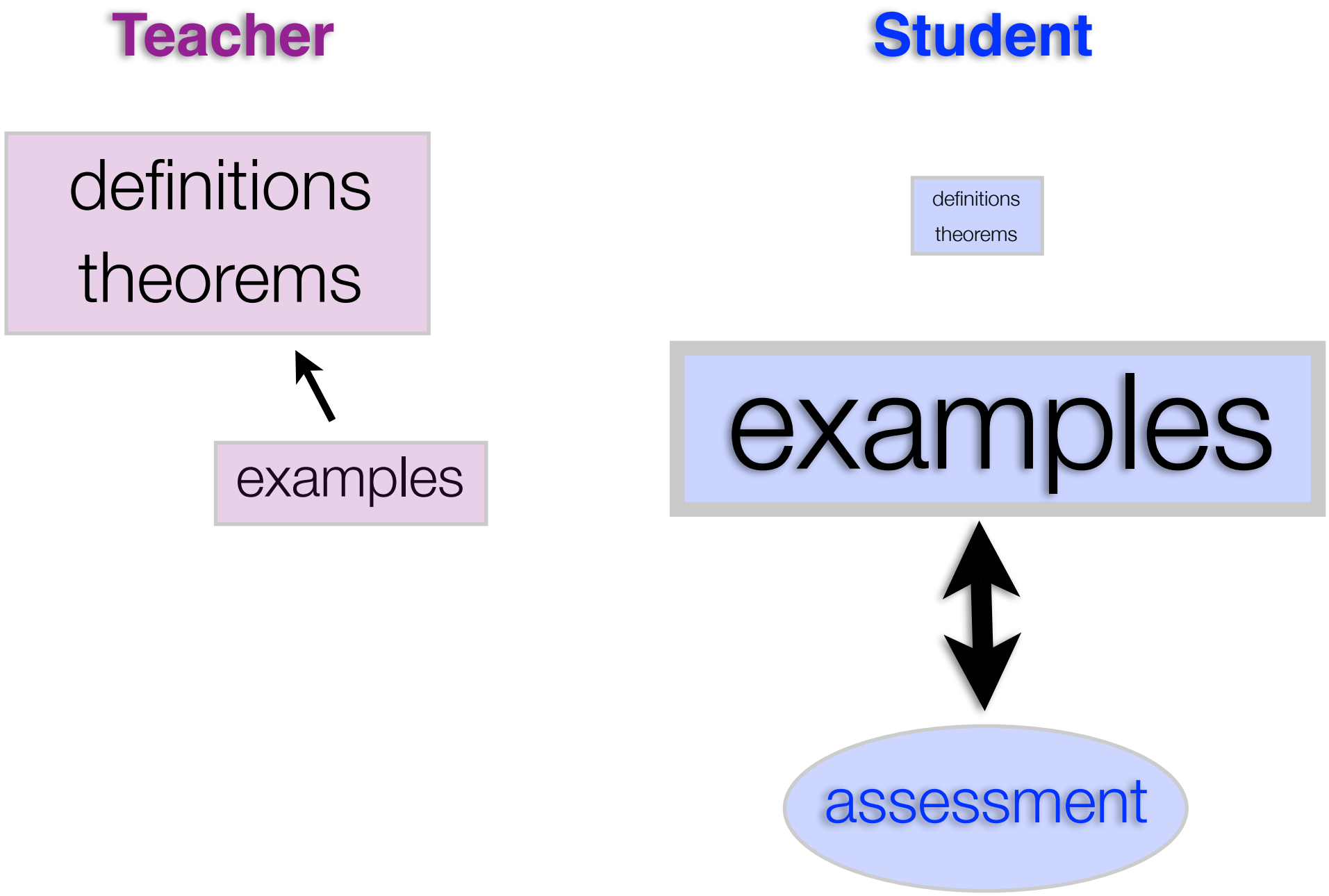

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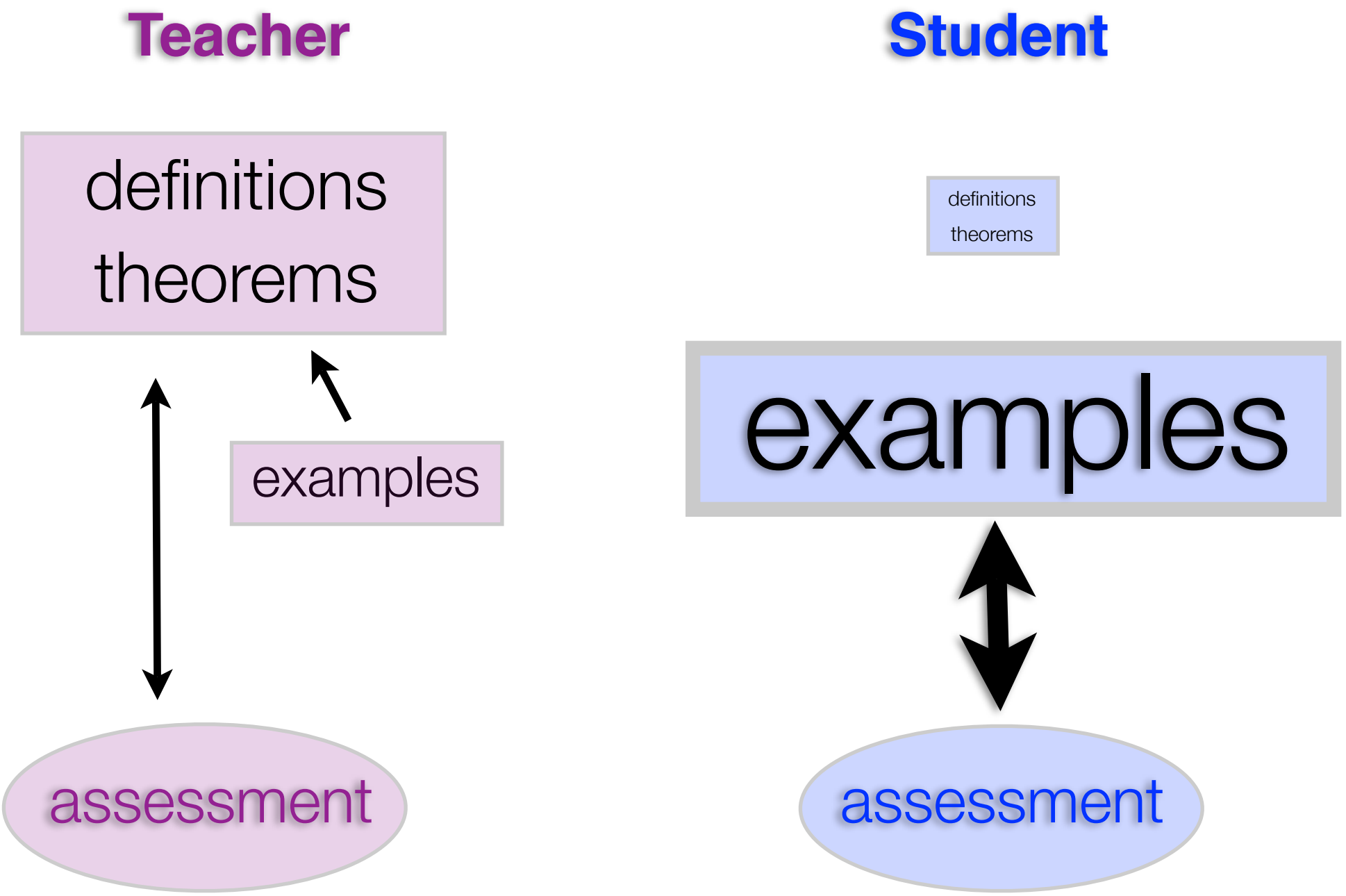

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Students can't think abstractly
can't do proofs
can't write.

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- Existence and definitions: existence proofs, unique existence.

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- TAs in ex. classes (4 h/w). One-to-one help is offered.

Essential dictionary

Sets: element, subset, empty set, union, intersection, difference, Cartesian product, power set, partition.

Numbers: natural numbers, integers, divisibility, primes, rational numbers, real numbers, complex numbers.

Functions: domain, co-domain, variable, value, image, inverse image, inverse, composition, restriction.

Sequences: general term, subscript, subsequence, vector.

Polynomials: indeterminate, monomial, coefficient, degree.

Equations: solution, identity, simultaneous equations.

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Difficult to teach

From symbols to words

Exercise 2.5. For each item, provide two levels of description: (i) a coarse description, which only identifies the class to which the item belongs (set, function, polynomial); (ii) a finer description, which defines the object in question, or characterises its structure (cf. example 2.6.2). **[$\not\in$]**

1. $3^3+4^3+5^3=6^3$
2. $\frac{7}{5}<\sqrt{2}<\frac{17}{12}$
3. x^3-x-1
4. $x^2-y^2=(x+y)(x-y)$
5. $xy>0$
6. $y-x^2-x=0$
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From words to symbols

Exercise 2.3. The following expressions define sets. Turn words into symbols, using standard or Zermelo definitions.
[*Represent geometrical objects, e.g., planar curves, by their cartesian equations.*]

1. The closed unit interval.
2. The set of negative odd integers.
3. The complement of the open unit disc in the plane.
4. The set of complex numbers that are not real.
5. The set of natural numbers with three decimal digits.
6. The set of vectors of unit length in three-dimensional euclidean space.
7. The set of circles in the plane, passing through the origin.
8. The set of hyperbolae in the cartesian plane, whose asymptotes are the cartesian coordinate axes.
9. The set of lines tangent to the unit circle. [*Think about it.*]

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$\left(\sum_{n=1}^{\infty} a_n\right)^2 \quad a_n \in \mathbb{Q}$ *the square of the sum of the elements of a rational sequence,*

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The set of divisors of an odd integer.

The set of polynomials over a field.

A proper infinite subset of the unit circle.

The Minkowski sum of two finite sets of integers.

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Describing functions

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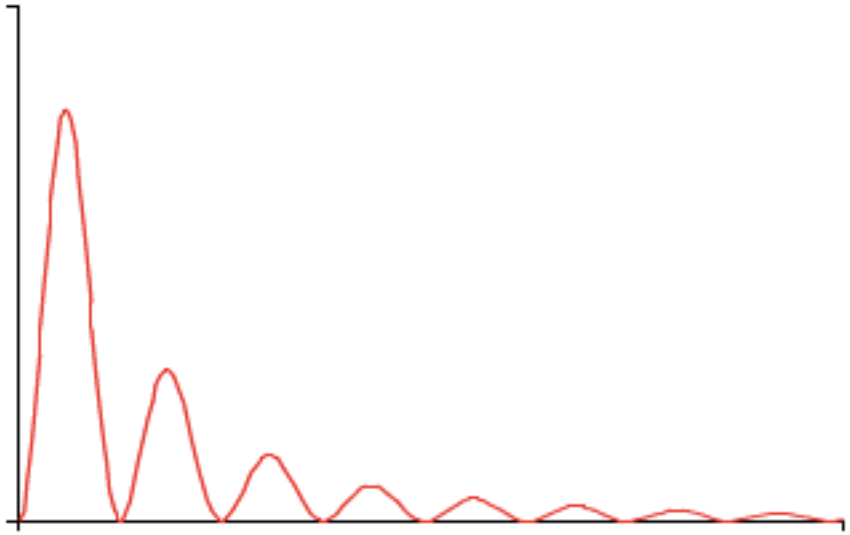

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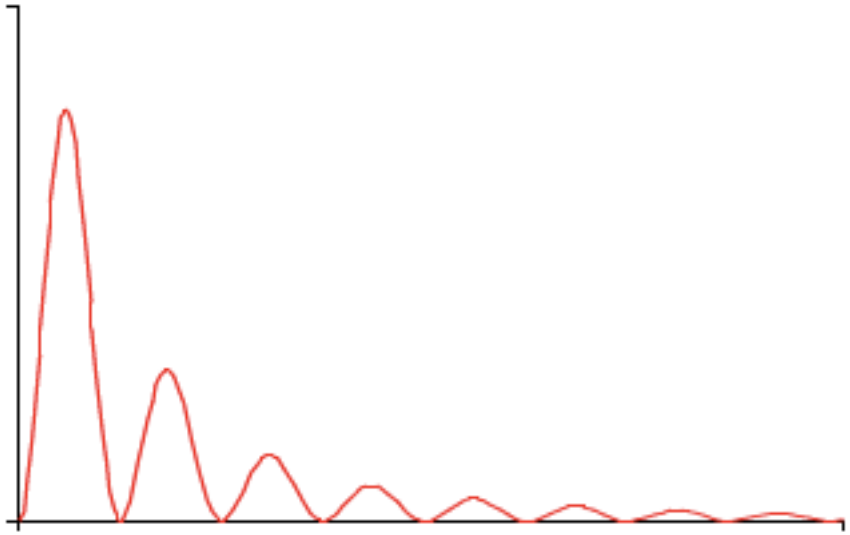

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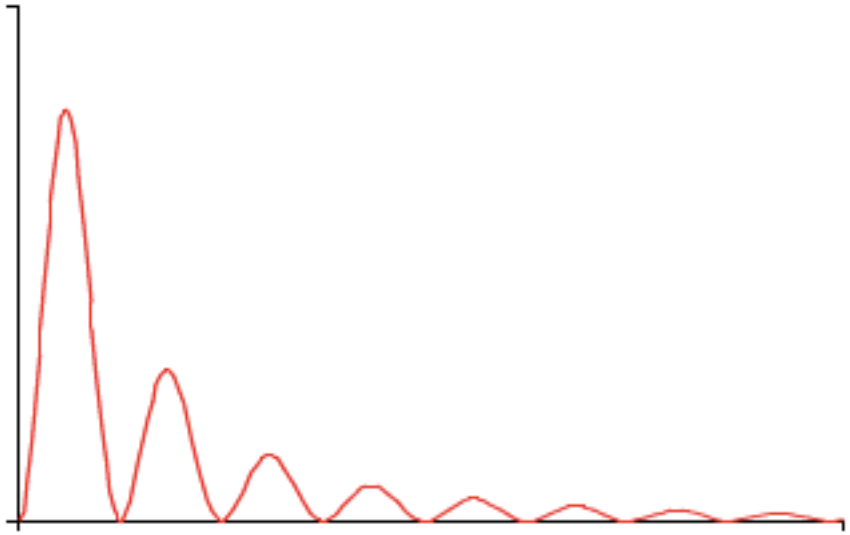

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- Universities should invest in centrally run schemes, to raise the profile of writing, and to support departmental efforts.

Thank you for your attention