Teaching how to think like a mathematician: writing mathematics

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Abstract
Writing well is an important transferable skill that is a traditional weakness for mathematics students. I explain how I approach teaching writing that encourages mathematical thinking. This has been successful and is something that any lecturer can attempt.

Background

A common myth is that mathematics teaches critical thinking, or in another variant, logical thinking. The evidence for this seems rather thin. An argument often advanced by lecturers is that they learned logical thinking by 'osmosis' from doing mathematics and so it is sufficient to expose students to mathematics to ensure the same. Experience shows, however, that exposure to mathematics does not necessarily produce a magical transformation of students into logical thinkers. This is perhaps not surprising as lecturers are atypical students - they became research mathematicians. The overwhelming majority of students, even at the highest institutions, do not become mathematics lecturers.

My approach to teaching critical thinking can be seen in my book *How to Think Like a Mathematician* [1]. Students are encouraged to ask themselves questions as they study. For example, given a statement, they should ask "What is the converse?" and "Is it true?" and "Where are the assumptions used in the proof?" These provide useful starting points for understanding.

The first step to logical thinking is not to give students a course in propositional logic ($p \wedge q \rightarrow p$ and all that) but to ensure that they write mathematics correctly. This may seem counterintuitive. Should we not ensure that they understand first and in later years insist that they write clearly? I believe not. Understanding and writing go hand in hand and good writing leads to an improvement in thinking.

There are a number of reasons to concentrate on writing:

i. Writing is a transferable skill.
ii. Improving writing skills eases the transition to university.
iii. To write clearly, students need to think clearly.

A good university education prepares a student for life and does not just fill their head with the facts of the subject. Writing well is an extremely useful skill for students as their chosen careers will likely involve producing written documents and arguing a case. Many students choose to study mathematics precisely because they do not like writing essays. Teaching writing is therefore essential. On the positive side, even a small amount of effort in this area can lead to a major boost for an individual student.

Improving the writing skills of students eases the transition to university. This is because university level mathematics is much more about understanding and creating coherent arguments whereas pre-university level is more about techniques, for example, finding the derivative of a function. Furthermore, insisting that students write clearly signals to them that the structure of arguments is important, just getting the "right" answer is insufficient.

Point (iii) above is crucial to the view that writing develops logical thinking. Obviously, if a student is not thinking clearly about mathematics, then they do not write clearly. A major effect of focussing on writing is that they are forced to think very carefully about how they are expressing themselves and by extension they have to think very carefully about the mathematics behind it. Also, if we allow students to submit a jumble of symbols without an argument, then we send the message that the numerical answer, say, is the important feature and the argument is not. Hence, they will focus on getting the "right" answer rather than making their arguments logical.

Implementation

The University of Leeds is a traditional red brick university in the north of England. The School of Mathematics had an intake in 2009 of 158 Home/EU students and 24 international students (the majority from China). Almost all the home students have A at A level in Mathematics and many have Further Mathematics at A or AS level.

During induction week students are given a booklet called How to Write Mathematics (HTWM) containing the two chapters on writing mathematics from my book and I give a twenty minute talk explaining the important points. A pdf file is freely available at <http://www.maths.leeds.ac.uk/~khouston/pdffiles/htwm.pdf >.

In their first year, standard BSc/MMath students have weekly tutorials in groups of 6 with pure, applied and statistics tutors (a mix of lecturers and PhD students). Students submit work weekly to their tutors for marking.

The HTWM booklet contains many examples of good writing practice. I tell my tutees that they should begin by concentrating on the following principles.

i. Write in sentences.
ii. Do not abuse the implication symbol.
iii. Equals means equals.
iv. Explain what is an assumption and what is a deduction.
v. Demonstrate that an equation is true by working on the more complicated side and showing that it is the same as the other side

Point (i) may seem obvious but many students do not see the need for sentences. Admittedly, until university, they usually could submit a collection of unconnected symbols and still achieve full marks.

The second point about abuse of the implication symbol is very important as it sends a signal to students about logical precision. Bertrand Russell said that mathematics is about P⇒Q (but you never ask whether P or Q is true). The starting point for logical thinking is to get ⇒ right and so we should be merciless in punishing its misuse. This poor symbol is one of the most overused by students and is pressed into service where it should not be. The belief is that using lots of ⇒ symbols makes an argument more mathematical. Common problems include using it at the start of sentence -- many students view it as saying "This is what we do next" -- and using it in place of the equals sign. For example, $\cos(\pi/6) + i \sin(\pi/6) \Rightarrow \sqrt{3}/2 + i/2$.

This brings us to (iii). Students do not always see the equals sign as meaning equal! They see no problem with having a vector on one side of an equation and a number on the other. Another example is induction. Suppose that $P(n)$ is the statement that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Students will happily write "Assume $P(n)$ is true, then $P(n)= \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$" which betrays that they do not distinguish the statement from the equation. This is of course a subtle point but by being firm in observing that "equals means equals" we force them into the pedantry necessary for mathematics. Another particular problem is that students produce errors by reaching for their calculators and using the resulting approximations in their calculations. Hence, I punish errors like $\sqrt{2}=1.41$.

To stop students seeing mathematics as being exclusively concerned with formulae and equations I tell students to explain what they are doing. For example, explaining what is an assumption and what is a conclusion. In the example from the How to Write Mathematics chapters mentioned earlier, rather than write "$a^2 = (c+x)^2 + h^2$, $a^2 = c^2 + 2cx + h^2 + x^2$, $b^2 = x^2 + h^2$, $a^2 = c^2 + 2cx + b^2$", they should, for example, write "From the diagram we see by Pythagoras' Theorem that $a^2 = (c+x)^2 + h^2 = c^2 + 2cx + h^2 + x^2$ and $b^2 = x^2 + h^2$. By substituting the second equation into the first we deduce that $a^2 = c^2 + 2cx + b^2$." The former is just a collection of symbols. The second is an explanation.

Furthermore, novice students generally do not know that in a proof we should use all the assumptions in the statement (or else the assumption was unnecessary). Drawing their attention to assumptions and deductions in their own work helps solve this problem.

Point (v) is useful as it makes proofs clearer and saves them writing. It also improves mathematical thinking. When showing an equation holds, students often start with the equation and manipulate it until they produce a true statement such as 1=1. As mathematicians we know that assuming what had to be proved is not correct. Insisting they show an equation holds by taking one side and simplifying it we do two things. First, we stop them making logical mistakes. Their usual method can be used to "prove" -1=1: "We have -1=1 implies that $(-1)^2=1^2$ by squaring both sides, since 1=1 the original equation is true!" Second, a common problem for students is that they do not know where to start on a problem. By taking only one side they have simplified their task since they have fewer expressions to work with.

Barriers
Over the years problems were encountered.

(a) A lot of initial effort is required as very detailed comments have to be made on students' work. However, this pays off later in the term, their work is much easier to read (and therefore less exhausting!).

(b) I often have to write the same comments repeatedly week after week, which can be dispiriting. I ease the burden by referring to the relevant section and page in the HTWM booklet.

(c) Students are against writing correctly. They often say things like "I didn't come to university to write essays", "But I got the right answer" or "You know what I meant". Dealing with this requires tact and patience as it takes time for the students to realize that the writing is helping them.

(d) Students don't read the feedback. An anecdote shows the problem and an admittedly extreme but successful solution. One of my tutees was apparently not reading my comments as he regularly made the same mistake. To remedy this I wrote on his work that if he did it again, then I would stamp on his work. Sure enough he did not read the comment, made the mistake again so I repeatedly stamped on his work in front of him. He was most surprised and of course had to admit that he never read my comments. His reason was that the numerical mark was the important thing. He also revealed that he knew that by reading the comments he could learn from his mistakes. This is a general problem. Students *do* know what is good for them. They know they should get up for a 9am lecture and they shouldn't leave revision until the night before. However human nature is against them.

Another answer to this is to not give a mark. This has problems in itself - students want a mark or else they can't judge how well they are doing. (In the UK the National Student Survey asks about quality of feedback and students may view the absence of a numerical grade as poor feedback.)

Enablers
As stated earlier the School of Mathematics gives a copy of the two chapters on writing from my book to the students during induction week. Also, the School trains PhD students in marking work, part of which involves the correct writing of mathematics. This means that students see that the School believes in the importance of writing and it is not just the hobbyhorse of one pedantic lecturer.

Evidence of success and impact

It is difficult to objectively measure the success of teaching writing to improve logical thinking as we never know what a student would have achieved without it. However, I have found that students do become more pedantic and start to see the details of the concepts they are describing. The whole area of writing is under-researched. I suspect that despite its importance it is ignored by lecturers. One solution I heard proposed to the problem of poor writing was "Tell students to write their work so that someone else can understand it". If only it were that simple.

Significant improvements in student writing are easier to see. I make copies of students' submitted work and a huge difference, and hence an impact, is evident by the end of the year.

There is evidence that the lessons learned are retained. While marking an anonymised second year exam I noticed that a weaker student was writing clearly. A later check confirmed my suspicion that the student was an ex-tutee.

Another important measure of success is that although students initially oppose writing clearly, they eventually change their mind and accept its necessity. Quite a number thank me later and say they hated it initially but are glad that they were forced to do it. Various reasons are given, including that it helped clarify for them what they did not understand. Enforcing clear writing is a good example illustrating that we should not give students what they want but give them what they didn't know they wanted.

Recommendations for others
First, I would not advocate introducing a module on writing mathematics. One, there is usually not space in the working week. Two, as in most education, skills should be integrated into existing modules or else they do not transfer well.

I am fortunate that I have only 6 students in a group. For modules with a large number of students it may be impractical to mark all the work to the necessary standard. One can save time by specifying the page number from the booklet. Another method is to produce a handout for all students discussing common problems. This obviously does not deal with problems specific to individuals but has the advantage that students see the mistakes of others.

It is important to be explicit that marks will be gained for good writing. Students are motivated by assessment and will pay little attention to something that does not get marks. We can send powerful signals to students about a subject by attaching marks to it. There is no harm however in pointing out that good writing skills will help them in their careers.

Often when students finally accept the importance of good writing they go to the extreme of writing out everything in great detail and their work becomes very long and unwieldy to mark. This is a good sign. It means they have grasped the importance of writing. Fortunately, it is very easy to rein them in -- they are relieved to write less!

Encouraging good writing requires persistence. I often repeat myself many times and even when the students finally understand, they sometimes return to their old ways after a few weeks. On the other hand, when a student has really "got it", unlike knowing some obscure mathematical fact, this is a skill they can apply in their lives long after they have left my charge. That is surely an activity worth doing.

[1] Houston K (2009)How to Think Like a Mathematician: A Companion to Undergraduate Mathematics, Kevin Houston, Cambridge University Press.