If Δz is taken so small at a given point z that $z+h$ lies in $\mathfrak{A}$, $\bar{g}$ is $\neq 0$. Next we note that

$$\lim_{\Delta z=0} g = \lim_{h=0} g(z+h) = g(z),$$

since g is continuous by 3. Passing now to the limit $\Delta z = 0$ in 6), we get 5).

5. By the aid of the foregoing we can find the derivative of a rational integral function

$$f = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m.$$

For as in the calculus we show that

$$\frac{dz^n}{dz} = nz^{n-1}.$$

Thus by 3),

$$\frac{df}{dz} = a_1 + 2\,a_2 z + \cdots + m a_m z^{n-1}.$$

Also the derivative of a rational function

$$k = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n} = \frac{f}{g}$$

can be found by 5).

6. Let us prove here a theorem we shall need later.

If $w = f(z)$ has a differential coefficient $f'(a) \neq 0$ at $x = a$, there exists a $\delta > 0$ such that Δw does not vanish when $z = a + \Delta z$ is restricted to $D_\delta^*(a)$.

For as

$$\lim_{\Delta z=0} \frac{\Delta w}{\Delta z} = f'(a) \quad , \quad \text{at } z = a,$$

we have

$$\Delta w = \{f'(a) + \epsilon'\}\,\Delta z \tag{7}$$

where $|\epsilon'| < \epsilon$ if only $0 < |\Delta z| <$ some δ.

If now we take $0 < \epsilon < |f'(a)|$

we see that $f'(a) + \epsilon'$ cannot vanish when $0 < |\Delta z| < \delta$. Thus Δw cannot vanish under this restriction, as 7) shows.

85. The Derivative of a Power Series. 1. Let the power series

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots \tag{1}$$

have $\mathfrak{C}$ as a circle of convergence. We show that P has a derivative within $\mathfrak{C}$, viz.:

$$P'(z) = a_1 + 2\,a_2 z + 3\,a_3 z^2 + \cdots \tag{2}$$

For by 43, 5)

$$P(z+h) = P(z) + hP_1(z) + \frac{1}{2\,!}h^2 P_2(z) + \cdots \tag{3}$$

where

$$P_1 = a_1 + 2\,a_2 z + 3\,a_3 z^2 + \cdots \tag{4}$$

which is the series on the right side of 2). As z is an arbitrary but fixed point, let us write 3)

$$P(z+h) = b_0 + b_1 h + b_2 h^2 + \cdots \tag{5}$$

This converges absolutely as long as the point $z+h$ lies within $\mathfrak{C}$, that is as long as

$$\eta = |h| \leq \text{some } \delta.$$

The adjoint of 5) is

$$\mathfrak{P} = \beta_0 + \beta_1 \eta + \beta_2 \eta^2 + \cdots$$

and as this converges for $\eta = \delta$,

$$\beta_0 + \beta_1 \delta + \beta_2 \delta^2 + \cdots = \beta_0 + \beta_1 \delta + \delta^2 \{\beta_2 + \beta_3 \delta + \beta_4 \delta^2 + \cdots\}$$

is convergent. Hence

$$\mathfrak{Q} = \beta_2 + \beta_3 \delta + \beta_4 \delta^2 + \cdots \tag{6}$$

is convergent.

From 3) and 5) we have

$$\frac{\Delta P}{\Delta z} = \frac{P(z+h) - P(z)}{h} = P_1(z) + h\{b_2 + b_3 h + \cdots\}$$
$$= P_1(z) + hQ. \tag{7}$$

Now each term of

$$Q = b_2 + b_3 h + b_4 h^2 + \cdots$$

is numerically $\leq$ the corresponding term of the series 6) when $|h| < \delta$. Thus

$$|Q| < \mathfrak{Q}, \text{ a constant.}$$

Hence $hQ \doteq 0$ as $h \doteq 0$. Hence, passing to the limit $h = 0$ in 7), we get 2). We have thus this result:

The function of z defined by a power series 1) *has a derivative within its circle of convergence, which is obtained by differentiating* 1) *term by term.*