The m^{th} row has here the sum

$$r_m = e^{-ma}.$$

Thus summing A by rows we get

$$R = r_1 + r_2 + \cdots$$
$$= e^{-a} + e^{-2a} + e^{-3a} + \cdots$$

This is a geometric series and converges absolutely since $a > 0$. We cannot infer, however, that A is convergent or that if it were its sum $= R$. In fact A is divergent. For if it were convergent each c_n series must be convergent by 2. This is not so, for

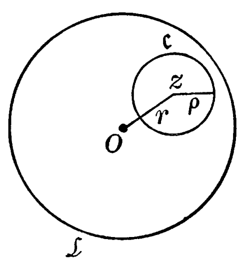
$$c_1 = 1 + 1 + 1 + \cdots$$

is divergent.

43. Application to Power Series. We wish to apply the foregoing theorem to obtain a result which we shall use later. Let the power series

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (1$$

have $\mathfrak{C}$ as a circle of convergence. About any point z within $\mathfrak{C}$ let us describe a circle $\mathfrak{c}$ of radius ρ which also lies within $\mathfrak{C}$. The point $z + h$ will lie in $\mathfrak{c}$ if $|h| \leq \rho$. Hence the series 1) converges absolutely when we replace z by $z + h$; that is



$$P(z + h) = a_0 + a_1(z + h) + a_2(z + h)^2 + \cdots \qquad (2$$

is an absolutely convergent series. Let us expand the terms of 2) and write the result as a double series. We get

$$\begin{aligned} A = a_0 &+ 0 + 0 + 0 + \cdots \\ + a_1 z &+ a_1 h + 0 + 0 + \cdots \\ + a_2 z^2 &+ 2\, a_2 z h + a_2 h^2 + 0 + \cdots \\ + a_3 z^3 &+ 3\, a_3 z^2 h + 3\, a_3 z h^2 + h^3 + \cdots \\ + \cdot &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{aligned} \qquad (3$$

If we sum 3) by rows, we get the absolutely convergent series 2). From this we cannot infer that 3) is convergent as we saw in 42, 4

The series A is, however, convergent, as we may easily see as follows. Let us set $|z| = r$. Then 1) converges absolutely for $z = r + \rho$ since this point lies within $\mathfrak{C}$. Thus

$$\begin{aligned}&\alpha_0 + \alpha_1(r + \rho) + \alpha_2(r + \rho)^2 + \cdots \\ &= \alpha_0 + \alpha_1 r + \alpha_1 \rho + \alpha_2 r^2 + 2\,\alpha_2 r\rho + \alpha_2 \rho^2 + \cdots\end{aligned} \tag{4}$$

is convergent. Thus the simple series

$$B = a_0 + a_1 z + a_1 h + a_1 z^2 + 2\,a_2 zh + a_2 h^2 + \cdots$$

is absolutely convergent since each of its terms is numerically $\geq$ the corresponding term of 4). Thus A is convergent and we can sum it by rows or by columns. Summing by rows gives

$$A = P(z+h).$$

Summing by columns we get, since the result is the same as before,

$$P(x + h) = P(z) + hP_1(z) + \frac{1}{2\,!}h^2P_2(z) + \frac{1}{3\,!}h^3P_3(z) + \cdots \quad (5$$

where

$$P_1 = a_1 + 2\,a_2 z + 3\,a_3 z + \cdots$$

$$P_2 = 2\,a_2 + 2 \cdot 3\,a_3 z + 3 \cdot 4\,a_4 z^2 + \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$